Equipartition of a Mass in Boxes

Siniša Vrećica

University of Belgrade

Topology and Combinatorics

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A problem

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- Could a mass in $\mathbb{R}^2$ be partitioned in 6 equal parts by three lines, two of which are parallel?
- When a mass in $\mathbb{R}^d$ admits an equipartition by $2k$ parallel hyperplanes and one additional hyperplane?

A problem

- More generally, when a mass in $\mathbb{R}^d$ could be equipartitioned by a collection of $2k$ parallel hyperplanes, and $m-1$ additional hyperplanes in $(2k+1) \times 2 \times \cdots \times 2$ parts?

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- We formulate these questions for even number of parallel hyperplanes for technical reasons. This case is a little bit easier to treat.

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- It is convenient to treat this question using eithe the obstruction theory or the equivariant index theory.
- In this talk we suppose some hyperplanes to be parallel, and use the latter theory.

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- $X =$ configuration space $=$ space of all candidates
- $(Y, A) =$ test space with subspace
- The symmetry of the problem induces an action of some group G both on X and (Y, A), making them G-spaces.
- Test map $f : X \to (Y, A)$ which is a G-map.

Scheme, continued

- $x \in X$ is a solution to the problem iff $f(x) \in A$.

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- If we show that for every G-map f, $\operatorname{im} f \cap A \neq \emptyset$ the problem has a solution.
- Most often $Y = \mathbb{R}^n$ and $A = \{0\}$. The problem reduces to the question whether for a G-map $f : X \to \mathbb{R}^n$, the requirement $0 \in \operatorname{im} f$ has to be satisfied, or to the question if there is a G-map $f : X \to S^{n-1}$.

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- THEOREM: *Every mass distribution in* $\mathbb{R}^d$ *could be equipartitioned by* $2(d-1)$ *parallel hyperplanes, and one additional hyperplane in* $4d-2$ *parts.*

A reduction

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- The first unit vector determines $2(d-1)$ parallel oriented hyperplanes dissecting $\mathbb{R}^d$ in $2d-1$ strips containing the same amount of the measure, i.e. $\frac{1}{2d-1}$.
  The second unit vector determines an oriented hyperplane halving the measure.

A reduction

The measures of the obtained $4d - 2$ parts form a $2 \times (2d - 1)$ matrix of the form

$$\begin{pmatrix} \alpha_0 + \alpha_1 & \alpha_0 + \alpha_2 & \dots & \alpha_0 + \alpha_{2d-1} \\ \alpha_0 - \alpha_1 & \alpha_0 - \alpha_2 & \dots & \alpha_0 - \alpha_{2d-1} \end{pmatrix}$$

where $\alpha_0 = \frac{1}{2(2d-1)}$
and $\alpha_1 + \alpha_2 + \cdots + \alpha_{2d-1} = 0.$

A reduction

- So the test space $M_{2\times(2d-1)}$ is the space of matrices of the above form and it is isomorphic to the subspace $V_{2d-2} = \{\alpha \in \mathbb{R}^{2d-1} \mid \sum_i \alpha_i = 0\}$.

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- Let $\varphi : S^{d-1} \times S^{d-1} \longrightarrow M_{2\times(2d-1)}$ be a test map assigning to every pair of unit vectors the measures of the corresponding $2 \times (2d-1)$ parts.

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- The group $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ acts on both spaces and the map φ is equivariant.
- The first generator of G permutes the columns of the matrix in the reversed order, and the second generator permutes its two rows.

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- If we suppose that $0 \notin \operatorname{im} \varphi$, then we have an equivariant map $\varphi : S^{d-1} \times S^{d-1} \longrightarrow S(V)$.

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- We use the ideal-valued cohomological index theory by E. Faddel and S. Husseini to show that such a map could not exist.
- An equivariant map $\varphi : X \longrightarrow Y$ induces a commutative diagram in cohomology:

Index theory

$$\begin{tikzcd}
H_G^*(X) & & H_G^*(Y) \arrow[ll, "\tilde{\varphi}^*"'] \\
& H^*(BG) \arrow[ul, "p_1^*"] \arrow[ur, "p_2^*"'] &
\end{tikzcd}$$

Figure 1:

Index theory, continued

- The kernels of p_1^* and p_2^* are the ideals in $H^*(BG)$, and are called indices and denoted by $\mathrm{Ind}_G(X)$ and $\mathrm{Ind}_G(Y)$.

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- From the above commutative diagram it follows $\mathrm{Ind}_G(Y) \subseteq \mathrm{Ind}_G(X)$.
- If we could determine these ideals and show that this relation does not hold, the obtained contradiction would show that there is no equivariant map from X to Y.

Index theory, our case

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- Also, $\mathrm{Ind}_G(S^{d-1} \times S^{d-1}) = (x_1^d, x_2^d)$, the ideal generated by these two monomials.
- In order to determine the index of the unit sphere of the representation space V, we will present it as the sum of the 1-dimensional G-invariant representations.

Index theory, our case

V splits in $d-1$ pairs of 1-dimensional spaces formed by the matrices of the form:

$$\begin{pmatrix} \dots & \alpha & \dots & -2\alpha & \dots & \alpha & \dots \\ \dots & -\alpha & \dots & 2\alpha & \dots & -\alpha & \dots \end{pmatrix}$$

and the matrices of the form:

$$\begin{pmatrix} \dots & \alpha & \dots & 0 & \dots & -\alpha & \dots \\ \dots & -\alpha & \dots & 0 & \dots & \alpha & \dots \end{pmatrix}$$

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- The first generator of G acts by permuting the columns in the reverse order, and so it acts trivially on the first mentioned 1-dimensional subspace and antipodally on the second. The second generator acts by permuting two rows, and so it acts antipodally on both 1-dimensional subspaces of matrices.

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- All the summands of this polynomial belong to the ideal (x_1^d, x_2^d), except for the first one $x_1^{d-1}x_2^{d-1}$.
- So, $\mathrm{Ind}_G S(V) \not\subseteq \mathrm{Ind}_G(S^{d-1}\times S^{d-1})$, which completes the proof.

The case $m = 3$

THEOREM: *Let* $\mathbb{P}_3 = \sum_{\sigma\in S_3} x^4_{\sigma(1)} x^2_{\sigma(2)} x_{\sigma(3)}$ *be a Dickson polynomial over* $\mathbb{Z}/2$. *Then every measure in* $\mathbb{R}^d$ *admits an equipartition by a collection of* $2k$ *parallel hyperplanes and two additional non-parallel hyperplanes in* $(2k+1)\times 2\times 2$ *boxes if*

$$(x_2+x_3)\left(\frac{1}{x_1}\mathbb{P}_3\right)^k \notin (x_1^d, x_2^d, x_3^d).$$

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  The second and the third unit vector determine the oriented hyperplanes halving the measure.

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The measures of the obtained parts form a $2 \times 2 \times (2k+1)$ matrix whose i-th "slice" is of the form

$$\begin{pmatrix} \varrho + \alpha_i & \varrho + \beta_i \\ \varrho + \gamma_i & \varrho + \delta_i \end{pmatrix}$$

$(i = 1, 2, ..., 2k+1)$, where $\varrho = \frac{1}{4(2k+1)}$, and $\alpha_i + \beta_i + \gamma_i + \delta_i = 0$ for every $i = 1, 2, ..., 2k+1$. Also, $\sum_i (\alpha_i + \beta_i) = 0$ and $\sum_i (\alpha_i + \gamma_i) = 0$.

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- The group $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ acts on both spaces and the map φ is equivariant.

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- In the other case we have an equivariant map $\varphi : S^{d-1} \times S^{d-1} \times S^{d-1} \longrightarrow S(V)$.

The case $m = 3$

- We have $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$, and so $H^*(BG; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, x_2, x_3]$.

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- So, the index $\mathrm{Ind}_G S(V)$ is the ideal generated by the polynomial $(x_2(x_1+x_2)x_3(x_1+x_3)(x_2+x_3)(x_1+x_2+x_3))^k(x_2+x_3) = (x_2+x_3)\left(\frac{1}{x_1}\mathbb{P}_3\right)^k$, and the result follows.

The corollaries

- COROLLARY: *Any mass distribution in* $\mathbb{R}^4$ *could be equipartitioned in* $3 \times 2 \times 2$ *boxes by a collection of* 4 *hyperplanes, two of which are parallel.*

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- Corollary: *Any mass distribution in* $\mathbb{R}^4$ *could be equipartitioned in* $3 \times 2 \times 2$ *boxes by a collection of* 4 *hyperplanes, two of which are parallel.*

- Corollary: *Any mass distribution in* $\mathbb{R}^8$ *could be equipartitioned in* $7 \times 2 \times 2$ *boxes by a collection of* 6 *parallel hyperplanes and two additional non-parallel hyperplanes.*

The general case

THEOREM: *Let* $\mathbb{P}_m = \sum_{\sigma \in S_m} x_{\sigma(1)}^{2^{m-1}} x_{\sigma(2)}^{2^{m-2}} \cdots x_{\sigma(m)}$ *be a Dickson polynomial over* $\mathbb{Z}/2$. *Then every measure in* $\mathbb{R}^d$ *admits an equipartition by a collection of* $2k$ *parallel hyperplanes and* $m-1$ *additional non-parallel hyperplanes in* $(2k+1) \times 2 \times \cdots \times 2$ *boxes if*

$$\frac{1}{x_2 x_3 \cdots x_m} \mathbb{P}_{m-1} \left(\frac{1}{x_1} \mathbb{P}_m \right)^k \notin (x_1^d, x_2^d, \ldots, x_m^d).$$

Limitations of the method

Our method does not provide the answer to the case when we consider the collections of parallel hyperplanes in 2 or more directions. Namely, there are infinitely many fixed points of the action of the group G on the test space in these cases, and so the equivariant map exists. The same is true if we consider the case of more than one mass distribution.

THANK YOU FOR YOUR ATTENTION!