

Cycle-free chessboard complexes and symmetric homology of algebras

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March 2008

Abstract

Chessboard complexes and their relatives have been an important recurring theme of topological combinatorics (see [1], [5], [10], [11], [13], [14], [20], [23], [24], [25]). Closely related “cycle-free chessboard complexes” have been recently introduced by Ault and Fiedorowicz in [2] and [9] as a tool for computing symmetric analogues of the cyclic homology of algebras. We study connectivity properties of these complexes and prove a result that confirms a strengthened conjecture from [2].

1 Introduction

Chessboard complexes and their relatives are well studied objects of topological combinatorics with applications in group theory, representation theory, commutative algebra, Lie theory, computational geometry, and combinatorics. The reader is referred to [13] and [23] for surveys and to [14], [20] for a guide to some of the latest developments.

Chessboard complexes originally appeared in [11] as coset complexes of the symmetric group, closely related to Coxeter and Tits coset complexes. After that they have been rediscovered several times. Among their avatars are “complexes of partial injective functions” [25], “multiple deleted joins” of 0-dimensional complexes [25] (implicit in [19]), the complex of all partial matchings in a complete bipartite graph, the complex of all non-taking rook configurations [5] etc.

Recently a naturally defined subcomplex of the chessboard complex, here referred to as “cycle-free chessboard complex”, has emerged in the context of stable homotopy theory ([2] and [9]). Ault and Fiedorowicz introduced this complex and its suspension $Sym_*^{(p)}$ as a tool for evaluating the symmetric analogue for the cyclic homology of

The authors are supported by Grants 144026 and 144014 of the Ministry for Science of Serbia.

algebras, [17]. They conjectured that $H_i\left(\text{Sym}_*^{(p)}\right) = 0$ if $i < p/2$, and verified this conjecture for some small values of p and i .

In this paper we prove this conjecture (Theorem 10) by showing that $\text{Sym}_*^{(p)}$ is actually γ_p -connected where $\gamma_p = \left\lfloor \frac{2}{3}(p-1) \right\rfloor$ (see Corollary 11). We also show (Theorem 13) that this result cannot be improved if $p = 3k + 2$ for some k and give evidence that the bound should be tight in the general case.

1.1 Graph complexes

Chessboard complexes $\Delta_{m,n}$ and their relatives are examples of *graph complexes*. A graph (digraph, multigraph) complex is (in *topological combinatorics*) a family of graphs (digraphs, multigraphs) on a given vertex set, closed under deletion of edges. Monograph [13], based on the author's Ph.D thesis [12], serves as an excellent source of information about applications of graph complexes in algebraic and geometric/topological combinatorics and related fields.

The appearance of a monograph solely devoted to the exposition and classification of simplicial complexes of graphs is probably a good sign of a relative maturity of the field. After decades of development, some of the central research themes and associated classes of examples have been isolated and explored, the technique is codified and typical applications described.

However, the appearance of a relative of the chessboard complex in the context of symmetric homology $HS_*(A)$ of algebras is perhaps of somewhat non-standard nature and deserves a comment.

Ault and Fiedorowicz showed in [2] (Theorem 6) that there exists a spectral sequence converging strongly to $HS_*(A)$ with the E^1 -term

$$E_{p,q}^1 = \bigoplus_{\bar{u} \in X^{p+1}/S_{p+1}} \tilde{H}_{p+q}(EG_{\bar{u}} \times_{G_{\bar{u}}} N\mathcal{S}_p/N\mathcal{S}'_p; k). \quad (1)$$

They emphasized (loc. cit.) the importance of the problem of determining the homotopy type of the space $N\mathcal{S}_p/N\mathcal{S}'_p$ and introduced a much more economical complex $\text{Sym}_*^{(p)}$ which computes its homology.

The complex $\text{Sym}_*^{(p)}$ turned out to be isomorphic to the suspension $\Sigma(\Omega_{p+1})$ of a subcomplex Ω_{p+1} of the chessboard complex $\Delta_{p+1} = \Delta_{p+1,p+1}$, one of the well studied graph complexes!

It is interesting to compare this development with the appearance of *2-connected graph complexes* [22] in the computation of the E^1 -term of the main Vassiliev spectral sequence converging to the cohomology

$$H^i(\mathcal{K} \setminus \Sigma) \cong \bar{H}_{\omega-i-1}(\Sigma) \cong \bar{H}_{\omega-i-1}(\sigma) \quad (2)$$

of the space $\mathcal{K} \setminus \Sigma$ of non-singular knots in \mathbb{R}^n . This spectral sequence arises from a filtration $\sigma_1 \supset \sigma_2 \supset \dots$ of a simplicial resolution σ of the space (discriminant) Σ of singular knots in \mathcal{K} . As a tool for computing $E_{i,j}^1 = \bar{H}_{i+j}(\sigma_i \setminus \sigma_{i-1})$, Vassiliev [22]

introduced an auxiliary filtration of the space $\sigma_i \setminus \sigma_{i-1}$. Complexes of 2-connected graphs naturally appear in the description of the E^1 -term of the spectral sequence associated to the auxiliary filtration.

It appears, at least on the formal level, that cycle-free chessboard complexes Ω_n play the role, in the Ault and Fiedorowicz approach to symmetric homology, analogous to the role of 2-connected graph complexes in Vassiliev's approach to the homology of knot spaces.

The homotopy type of the complex of (not) 2-connected graphs was (independently) determined by Babson, Björner, Linusson, Shareshian, and Welker in [3] and Turchin in [21]. This development stimulated further study of connectivity of graph properties (complexes), see chapter VI of [13] ([12]).

2 Cycle-free chessboard complexes

Chessboard complexes $\Delta_{m,n}$ are matching (graph) complexes associated to complete bipartite graphs [13], [20], [23]. However, they most naturally arise as complexes of *admissible rook configurations* on general $m \times n$ chessboards.

An $(m \times n)$ -chessboard is the set $A_{m,n} = [m] \times [n] \subset \mathbb{Z}^2$ where (as usual in combinatorics) $[n] = \{1, 2, \dots, n\}$. The associated *chessboard complex* $\Delta(A_{m,n}) = \Delta_{m,n}$ is defined as the (abstract, simplicial) complex of all admissible or *non-taking rook configurations* on the chessboard $A_{m,n}$. More generally, for an arbitrary (finite) subset $A \subset \mathbb{Z}^2$, the associated chessboard complex $\Delta(A)$ has A for the set of vertices and $S \in \Delta(A)$ if and only if for each pair $(i, j), (i', j')$ of distinct vertices of S both $i \neq i'$ and $j \neq j'$. Also, we often denote by $\Delta_{X,Y} = \Delta(X \times Y)$ the chessboard complex carried by the "chessboard" $X \times Y$ where X and Y are not necessarily subsets of \mathbb{Z} .

Let $\Delta_n := \Delta_{n,n}$ be the chessboard complex associated to the canonical $(n \times n)$ -chessboard $[n] \times [n]$, similarly $\Delta_X := \Delta_{X,X}$. Each top dimensional simplex in Δ_n is essentially the graph $\Gamma_\phi := \{(i, \phi(i)) \mid i \in [n]\}$ of a permutation $\phi : [n] \rightarrow [n]$. Any other simplex $S \in \Delta_n$ arises as a top dimensional simplex of the complex $\Delta(A \times B)$ where A and B are two subsets of $[n]$ of equal size. Alternatively S can be described as the graph Γ_ψ of a bijection $\psi : A \rightarrow B$, which is sometimes referred to as a partial, injective function (relation) defined on $[n]$.

Definition 1. *A non-taking rook configuration $S \subset [n] \times [n]$ of size $n - 1$ is cycle-free if there is a linear order $\rho : i_1 \prec i_2 \prec \dots \prec i_n$ of elements of the set $[n]$ such that*

$$S = S_\rho = \{(i_1, i_2), (i_2, i_3), \dots, (i_{n-1}, i_n)\}.$$

Define $\Omega_n := \bigcup_{\rho \in LO_n} S_\rho \subset \Delta_n$ as the union, or \subset -ideal closure, of the collection of all cycle-free configurations S_ρ , $\rho \in LO_n$, where LO_n is the set of all linear orders on $[n]$.

Alternatively the complex Ω_n can be described as the collection of all non-taking rook placements $S \in \Delta_n$ which do not contain cycles, that is sub-configurations of the form $\{(x_1, x_2), (x_2, x_3), \dots, (x_m, x_1)\}$ for some $1 \leq m \leq n$. For this reason we call Ω_n the chessboard complex without cycles or simply the cycle-free chessboard complex.

In order to study functorial properties of complexes Ω_n it is convenient to extend slightly the definitions and introduce a class of more general cycle-free chessboard complexes. In Section 3 we introduce an even larger class of hybrid chessboard complexes which contain Ω_n and Δ_n , as well as $\Delta(A)$ for $A \subset \mathbb{Z}^2$, as special cases.

If X is a finite set then $\Omega_X \subset \Delta_X = \Delta_{X,X}$ is defined as the union of all simplices $S_\rho = \{(x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n)\}$ where $\rho : x_1 \prec x_2 \prec \dots \prec x_n$ is a linear order on X . More generally, given a bijection $\alpha : Y \rightarrow X$ of two finite sets, let $\Omega(X \times Y; \alpha)$ be the complex of all non-taking rook configurations in $X \times Y$ without sub-configurations of the form $\{(x_1, y_2), (x_2, y_3), \dots, (x_m, y_1)\}$ where $1 \leq m \leq |X| = n$ and $x_j = \alpha(y_j)$ for each j . It is clear that all these complexes are isomorphic to Ω_n if $|X| = |Y| = n$.

For visualization and as a convenient bookkeeping device, simplices in $\Delta(X \times Y)$ as well as in $\Omega(X \times Y; \alpha)$ can be represented as matchings in the complete bipartite graph $K_{X,Y}$.

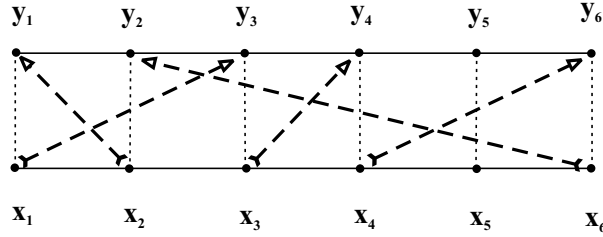


Figure 1: A cycle in Δ_6 .

The partial matching $\{(x_1, y_3), (x_2, y_1), (x_3, y_4), (x_4, y_6), (x_6, y_2)\}$, exhibited in Figure 1, clearly determines a non-taking rook placement on the chessboard $X \times Y$ where $X = \{x_i\}_{i=1}^6$ and $Y = \{y_i\}_{i=1}^6$. If $\alpha : Y \rightarrow X$ is the bijection $y_j \mapsto x_j$ then this matching does not contribute a simplex to $\Omega(X \times Y; \alpha)$ since it contains a cycle

$$x_1 \mapsto y_3 \downarrow x_3 \mapsto y_4 \downarrow x_4 \mapsto y_6 \downarrow x_6 \mapsto y_2 \downarrow x_2 \mapsto y_1.$$

The following proposition establishes a key structural property for cycle-free chessboard complexes Ω_n .

Proposition 2. *The link $\text{Link}(v) = \text{Link}_{\Omega_n}(v)$ of each vertex v in the cycle-free chessboard complex Ω_n is isomorphic to Ω_{n-1} .*

Proof: Let us choose $\Omega(X \times Y; \alpha)$ as our model for Ω_n where $X = \{x_i\}_{i=1}^n$ and $Y = \{y_i\}_{i=1}^n$ while the bijection $\alpha : Y \rightarrow X$ maps y_j to x_j . Let $v = (x_i, y_j) \in X \times Y$, where $i \neq j$.

Define $X' := X \setminus \{x_i\}$ and $Y' := Y \setminus \{y_j\}$. Let $\alpha' : Y' \rightarrow X'$ be the bijection defined by $\alpha'(y_i) = x_j$ and $\alpha'(y_k) = x_k$ for $k \in [n] \setminus \{i, j\}$. Then it is not difficult to show that $\text{Link}_{\Omega_n}(v) \cong \Omega(X' \times Y', \alpha') \cong \Omega_{n-1}$. \square

2.1 Ω_n as a digraph complex

Chessboard complexes Δ_n and cycle-free chessboard complexes Ω_n , as well as their natural generalizations, admit another, equally useful description as directed graph (digraph) complexes.

A chessboard $A_n = [n] \times [n]$ is naturally interpreted as a complete digraph DK_n (loops included) where each $(i, j) \in A_n$ contributes a directed edge \vec{ij} in DK_n . A directed subgraph $\Gamma \subset DK_n$ describes an admissible rook configuration in A_n if and only if no two directed edges in Γ are allowed to have the same *tail* or the same *end*.

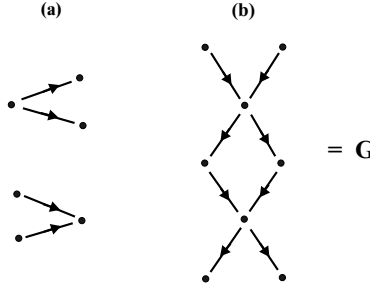


Figure 2: $\Omega(G) = S^0 * S^0 * S^0 * S^0 = S^3$.

In other words configurations depicted in Figure 2 (a) are banned from the graph Γ . It follows that Δ_n is the complex of all subgraphs of DK_n such that the associated connected components are either directed cycles or directed paths. The complex Ω_n arises as the cycle-free subcomplex of Δ_n , i.e. $\Gamma \in \Omega_n$ if only directed paths are allowed as connected components of Γ . This definition reveals that probably the closest relative of Ω_n , that has been systematically analyzed so far, is the complex Δ_n^{DM} of directed matchings on the node set $[n]$ introduced in [6].

More generally, for each directed graph G one can define the associated complexes $\Delta(G)$ and $\Omega(G)$ as the complexes of all directed subgraphs Γ in G which have only directed paths and cycles (respectively paths alone) as connected components. For example if G is the directed graph depicted in Figure 2 (b) then $\Delta(G) = \Omega(G) \cong S^3$.

3 Generalized cycle-free complexes

Let $\Omega(X \times Y, \alpha)$ be the cycle-free chessboard complex associated to sets $X, Y \subset \mathbb{Z}$ and a bijection $\alpha : Y \rightarrow X$. Assume that $A \subset \mathbb{Z}^2$ is a finite superset of $X \times Y$. Define $\Omega = \Omega(A, X \times Y, \alpha)$ as the subcomplex of the full chessboard complex $\Delta(A)$ by the condition that $S \in \Delta(A)$ is in Ω if and only if the restriction of S on $\Delta(X \times Y)$ is in $\Omega(X \times Y, \alpha)$. Ω is referred to as the generalized cycle-free chessboard complex.

If $A = (X \cup Z) \times (Y \cup T)$, where $X \cap Z = \emptyset = Y \cap T$, let

$$\Omega_{X,Z}^{Y,T} := \Omega(A, X \times Y, \alpha).$$

The isomorphism type of the complex $\Omega_{X,Z}^{Y,T}$ depends only on cardinalities of sets X, Y, Z, T so if $|X| = |Y| = n, |Z| = m$, and $|T| = p$, we will frequently denote by

$\Omega_{n,m}^{n,p}$ one of its unspecified representatives. If $p = 0$ we write $\Omega_{n,m} := \Omega_{n,m}^{n,0}$ and if $m = 0$, the complex $\Omega_{n,0} = \Omega_n$ reduces to the standard cycle-free chessboard complex defined on a $n \times n$ -chessboard.

Definition 3. Let $\Omega = \Omega(A, X \times Y, \alpha)$ be a generalized, cycle-free chessboard complex based on a chessboard $A \subset \mathbb{Z}^2$, where $X \times Y \subset A$ and $\alpha : Y \rightarrow X$ is an associated bijection. Let $v = (a, b) \in A$. The v -reduced complex $\Omega' = \Omega'_v = \Omega(A', X' \times Y', \alpha')$ of Ω is defined as follows. Let $A' := A \setminus (\{a\} \times \mathbb{Z} \cup \mathbb{Z} \times \{b\})$.

- (a) If both $a \in X$ and $b \in Y$ let $X' := X \setminus \{a\}$, $Y' := Y \setminus \{b\}$ and let $\alpha' : Y' \rightarrow X'$ be the bijection defined by $\alpha'(\alpha^{-1}(a)) := \alpha(b)$, and $\alpha'(z) = \alpha(z)$ for $z \neq \alpha^{-1}(a)$.
- (b) If $a \in X$ and $b \notin Y$ let $X' := X \setminus \{a\}$, $Y' := Y \setminus \{\alpha^{-1}(a)\}$ and $\alpha' : Y' \rightarrow X'$ is the restriction of α on Y' .
- (c) If $b \in Y$ and $a \notin X$ let $Y' := Y \setminus \{b\}$, $X' := X \setminus \{\alpha(b)\}$ and $\alpha' : Y' \rightarrow X'$ is the restriction of α on Y' .
- (d) If neither $a \in X$ nor $b \in Y$, let $X' = X$, $Y' = Y$ and $\alpha' = \alpha$.

The following proposition records for the future reference the key structural property of generalized cycle-free chessboard complexes $\Omega = \Omega(A, X \times Y, \alpha)$. The proof is similar to the proof of Proposition 2 so we omit the details.

Proposition 4. If $\text{Link}(v) = \text{Link}_\Omega(v)$ is the link of a vertex $v = (a, b) \in A$ in $\Omega = \Omega(A, X \times Y, \alpha)$ then there is an isomorphism

$$\text{Link}(v) \cong \Omega(A', X' \times Y', \alpha')$$

where $\Omega(A', X' \times Y', \alpha')$ is the v -reduced complex of the generalized cycle-free chessboard complex $\Omega(A, X \times Y, \alpha)$ (Definition 3).

4 Filtrations of chessboard complexes

The chessboard complex $\Delta(A)$ functorially depends on the chessboard $A \subset \mathbb{Z}^2$. It follows that a filtration

$$A_0 \subset A_1 \subset \dots \subset A_{m-1} \subset A_m \subset A$$

induces a filtration of the complex $\Delta(A)$,

$$\Delta(A_0) \subset \Delta(A_1) \subset \dots \subset \Delta(A_{m-1}) \subset \Delta(A_m) \subset \Delta(A).$$

This filtration in turn induces a filtration $\{F_j(\Omega)\}_{j=0}^m$ of the associated generalized, cycle-free chessboard complex $\Omega = \Omega(A, X \times Y, \alpha)$. If $X \times Y \subset A_0$ then clearly $F_j(\Omega) = \Omega(A_j, X \times Y, \alpha)$. We are particularly interested in filtrations where $A_j \setminus A_{j-1} = \{a_j\}$

is a singleton. Consequently a filtration is determined once we choose a linear order of the elements (elementary squares) of the set $A \setminus A_0$.

A basic fact and a well known consequence of the *Gluing Lemma* [8] is that the homotopy type of the “double mapping cylinder” (homotopy colimit) of the diagram $B \xleftarrow{f} A \xrightarrow{g} C$ of spaces (complexes) depends only on homotopy types of maps f and g . It follows that if both maps f and g are homotopic to constant maps the associated double mapping cylinder has the homotopy type of a wedge $B \vee \Sigma(A) \vee C$. From here we immediately deduce that if a simplicial complex $X = X_1 \cup X_2$ is expressed as a union of its sub-complexes such that both X_1 and X_2 have the homotopy type of a wedge of n -dimensional spheres while the intersection $X_1 \cap X_2$ is a wedge of $(n - 1)$ -dimensional spheres, then the complex X is also a wedge of n -dimensional spheres. An immediate consequence is the following lemma.

Lemma 5. *Let K be a finite simplicial complex. Given a vertex $v \in K$, let $\text{Link}_K(v)$ and $\text{Star}_K(v)$ be the link and star subcomplex of K . Let $A\text{-Star}_K(v) = K \setminus \{v\}$ be the “anti-star” of v in K , i.e. the complex obtained by deleting v from all simplices, or equivalently by removing the “open star” of v from K . If $A\text{-Star}(v)$ is homotopy equivalent to a wedge of n -dimensional spheres and $\text{Link}_K(v)$ is homotopy equivalent to a wedge of $(n - 1)$ -dimensional spheres, then the complex K itself has the homotopy type of a wedge of n -dimensional spheres.*

One way of proving that a simplicial complex is homotopically a wedge of n -spheres is to iterate Lemma 5. In the following section we show that among the complexes where this strategy can be successfully carried on are some generalized cycle-free complexes.

5 Complexes $\Omega_{n,m}$

Proposition 6. *The complex $\Omega_{n,m}$ is homotopy equivalent to a wedge of $(n - 1)$ -dimensional spheres provided $m \geq n$.*

Proof: Let us establish the statement for all complexes $\Omega_{n,m}$, where $m \geq n$, by induction on n . Note that $\Omega_{2,2}$ is a circle and that $\Omega_{2,m}$ for $m \geq 3$ is always a connected, 1-dimensional complex, hence a wedge of 1-spheres.

Assume, as an inductive hypothesis, that $\Omega_{n,m}$ is homotopic to a wedge of $(n - 1)$ -spheres for each $m \geq n$.

Our model for $\Omega_{n,n}$ will be the complex $\Omega_{X,Z}^Y$ where $X = \{x_i\}_{i=1}^n$, $Y = \{y_i\}_{i=1}^n$, $Z = \{z_i\}_{i=1}^n$, where $\alpha : Y \rightarrow X$ is the canonical bijection $y_j \mapsto x_j$.

Our model for $\Omega_{n+1,n+1}$ will be the complex $\Omega_{X',Z'}^{Y'}$ where $X' = X \cup \{x_0\}$, $Y' = Y \cup \{y_0\}$, $Z' = Z \cup \{z_0\}$ and the bijection $\alpha' : Y' \rightarrow X'$ is the (unique) extension of α characterized by $\alpha'(y_0) = x_0$.

Following the strategy outlined in Section 4, we define a filtration of the complex $\Omega_{n+1,n+1}$ by choosing $A_0 = \{(z_0, y_0)\} \cup ((X' \cup Z) \times Y)$ as the initial chessboard and selecting a linear order on the set $W := ((X \cup Z) \times \{y_0\}) \cup (\{z_0\} \times Y)$ of elementary

squares (Figure 3). Note that the element (x_0, y_0) is omitted since it is not allowed to be a vertex of the cycle-free complex $\Omega(X', Y')$. Let

$$P = \{(x_i, y_0)\}_{i=1}^n, \quad Q = \{(z_i, y_0)\}_{i=1}^n, \quad R = \{(z_0, y_i)\}_{i=1}^n.$$

List elements of $W = P \cup Q \cup R$ in the order of appearance in this \cup -decomposition. Within each of the blocks P, Q, R the elements can be ordered in an arbitrary way, say according to the index $i = 1, \dots, n$.

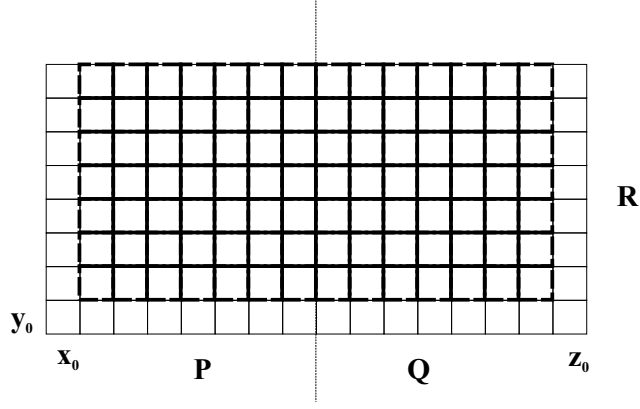


Figure 3:

If $W = \{v_k\}_{k=1}^N$ where $N = 3n$, let

$$\{(z_0, y_0)\} \cup ((X' \cup Z) \times Y) = A_0 \subset A_1 \subset \dots \subset A_N = A = (X' \cup Z') \times Y' \quad (3)$$

be the filtration defined by $A_j := A_0 \cup \{v_k\}_{k=1}^j$. Let $\{\Delta(A_j)\}_{j=0}^N$ be the associated filtration of the chessboard complex $\Delta(A)$ and let $\{F_j(\Omega)\}_{j=0}^N$ be the induced filtration on the generalized cycle-free complex $\Omega = \Omega(A, X' \times Y', \alpha')$. Note that $F_j(\Omega) = \Omega(A_j, X' \times Y', \alpha)$ for $j \geq n$ while in general $F_j(\Omega) = \Omega(A, X' \times Y', \alpha') \cap \Delta(A_j)$.

By Proposition 4, the homotopy type of the link $\text{Link}_k(v_k)$ of v_k in the complex $F_k(\Omega)$ can be described as follows.

(I) $v_k \in P$, i.e. $v_k = (x_i, y_0)$ for some $i = 1, \dots, n$.

$$\text{Link}_k(v_k) \cong \Omega_{n,n}$$

(II) $v_k \in Q$, i.e. $v_k = (z_i, y_0)$ for some $i = 1, \dots, n$.

$$\text{Link}_k(v_k) \cong \Omega_{n,n}.$$

(III) $v_k \in R$, i.e. $v_k = (z_0, y_i)$ for some $i = 1, \dots, n$.

$$\text{Link}_k(v_k) \cong \Omega_{n,n+1}.$$

The complex $F_0(\Omega)$ is a cone with apex (z_0, y_0) , hence it is contractible. In all cases (I)–(III), by the inductive hypothesis, the complexes $\Omega_{n,n}$ and $\Omega_{n,n+1}$ have the homotopy type of a wedge of $(n - 1)$ -dimensional spheres. Consequently, by repeated use of Lemma 5, $\Omega_{n+1,n+1}$ has the homotopy type of a wedge of n -dimensional spheres.

It remains to be shown that the complex $\Omega_{n+1,m}$ has the homotopy type of a wedge of n -dimensional spheres if $m > n + 1$. This is achieved by expanding the filtration (3) by adding vertices from new columns, in some order, and applying the same argument as above. \square

6 Complexes $\Omega_{n,m}$ and the nerve lemma

A classical result of topological combinatorics is the Nerve Lemma. It was originally proved by J. Leray in [16], see also [4] for a more recent overview of applications and related results.

Lemma 7. (Nerve Lemma, [16]) *Let Δ be a simplicial complex and $\{L_i\}_{i=1}^k$ a family of subcomplexes such that $\Delta = \cup_{i=1}^k L_i$. Suppose that every nonempty intersection $L_{i_1} \cap L_{i_2} \cap \dots \cap L_{i_t}$ is $(\mu - t + 1)$ -connected for $t \geq 1$. Then Δ is μ -connected if and only if $\mathcal{N}(\{L_i\}_{i=1}^k)$, the nerve of the covering $\{L_i\}_{i=1}^k$, is μ -connected.*

In the preceding section we showed that for $m \geq n$ the complex $\Omega_{n,m}$ is a wedge of $(n - 1)$ -dimensional spheres, consequently it is $(n - 2)$ -connected. Here we continue the analysis of these complexes and establish a lower bound for the connectivity of the complex $\Omega_{n,m}$ for any $m \geq 1$.

Proposition 8. *The complex $\Omega_{n,m}$ is $\mu_{n,m}$ -connected, where*

$$\mu_{n,m} = \min \left\{ \left\lfloor \frac{2n + m}{3} \right\rfloor - 2, n - 2 \right\}.$$

Proof: We proceed by induction on n . For $n = 2$, the complex $\Omega_{2,1}$ is the union of two segments and so non-empty (or (-1) -connected), and for $m \geq 2$ the complex $\Omega_{2,m}$ is clearly connected (or 0-connected).

Let us suppose that complexes $\Omega_{r,m}$ are $\mu_{r,m}$ -connected, whenever $r \leq n - 1$, and consider the complex $\Omega_{n,m}$. If $m \geq n$, then $\mu_{n,m} = n - 2$, and the complex $\Omega_{n,m}$ is $(n - 2)$ -connected by Proposition 6. Suppose that $1 \leq m \leq n - 1$, which implies that $\mu_{n,m} \leq n - 3$.

We use $\Omega_{[n],Z}^{[n],\emptyset}$, where $|Z| = m$, as a model for the complex $\Omega_{n,m}$. For example, in order to keep our chessboards in \mathbb{Z}^2 , we could take $Z = \{-1, -2, \dots, -m\}$. Let $\mathcal{L}_{n,m} = \{L_{z,i} \mid z \in Z, i \in [n]\}$ be the family of subcomplexes of $\Omega_{n,m}$ where by definition $L_{z,i} := \text{Star}((z, i))$ is the union of all simplices with (z, i) as a vertex, together with their faces. Every maximal simplex in $\Omega_{n,m}$ must have a vertex belonging to $Z \times [n]$. So, the collection $\mathcal{L}_{n,m}$ of contractible complexes is a covering of $\Omega_{n,m}$.

Let us apply the Nerve Lemma. It is easy to see that the intersections of any $n - 1$ complexes $L_{z,i}$ is nonempty. It follows that the nerve $\mathcal{N}(\mathcal{L}_{n,m})$ of the covering contains the full $(n - 2)$ -dimensional skeleton, hence it is at least $(n - 3)$ -connected. It remains to show that the intersection of any subcollection of t of these complexes is at least $(\mu_{n,m} - t + 1)$ -connected.

For the reader's convenience, we begin with the simplest case $t = 2$. There are three possibilities for the intersection $L_{z_1,i} \cap L_{z_2,j}$.

- If $z_1 \neq z_2$ and $i \neq j$, this intersection is a join of the interval spanned by vertices $(z_1, i), (z_2, j)$, and a subcomplex of type $\Omega_{n-2,m}$. Therefore, it is contractible.
- If $z_1 \neq z_2$ and $i = j$, this intersection is a subcomplex of type $\Omega_{n-1,m-1}$, which is at least $\mu_{n-1,m-1} = (\mu_{n,m} - 1)$ -connected by the induction hypothesis.
- If $z_1 = z_2$ and $i \neq j$, this intersection is a subcomplex of the type $\Omega_{n-2,m+1}$ which is $\mu_{n-2,m+1}$ -connected.

Then $\left\lceil \frac{2(n-2)+(m+1)}{3} \right\rceil - 2 = \mu_{n,m} - 1$. Also, $(n - 2) - 2 \geq \mu_{n,m} - 1$ because $\mu_{n,m} \leq n - 3$. Therefore, $\mu_{n-2,m+1} \geq \mu_{n,m} - 1$.

Similar arguments apply also in the case $t \geq 3$. The intersection $L_{z_1,i_1} \cap L_{z_2,i_2} \cap \dots \cap L_{z_t,i_t}$ could be either contractible (when for some $h \in \{1, 2, \dots, t\}$ both z_h and i_h are different from all other z_j and i_j respectively), or it could be a subcomplex of the type $\Omega_{r,s}$ where both $r \geq n - t$ and $r + s \geq n + m - t$. Then $2r + s \geq 2n + m - 2t \geq 2n + m - 3t + 3$. Actually it could be easily proved more, i.e. that $2r + s \geq 2n + m - \frac{3}{2}t$, but we need the more precise estimate only in the case $t = 2$.

The above inequality implies $\left\lceil \frac{2r+s}{3} \right\rceil - 2 \geq \mu_{n,m} - t + 1$. Also, $r - 2 \geq n - t - 2 \geq \mu_{n,m} - t + 1$, because $\mu_{n,m} \leq n - 3$.

These two facts together imply that $\mu_{r,s} = \min \left\{ \left\lceil \frac{2r+s}{3} \right\rceil - 2, r - 2 \right\} \geq \mu_{n,m} - t + 1$ which is precisely the desired inequality. \square

7 Complexes Ω_n

Now we are ready to prove our main result, i.e. to establish high-connectivity of the complex Ω_n .

Proposition 9. *For each $n \geq 5$, $\pi_1(\Omega_n) = 0$.*

Proof: We apply the Nerve Lemma on the complex $L := L_1 \cup L_2 \cup L_3$ where L_j is the subcomplex of Ω_n based on the chessboard $[n] \times ([n] \setminus \{i\})$. In other words a simplex $\sigma \in \Omega_n$ is in L_i if and only if it doesn't have a vertex of the type (\cdot, i) .

It is clear that the 1-skeleton of Ω_n is a subcomplex of L , hence it suffices to show that L is 1-connected. Since $L_1 \cap L_2 \cap L_3 \neq \emptyset$ it is sufficient to show that L_i is 1-connected for each i and that $L_i \cap L_j$ is connected for each pair $i \neq j$. Since $L_i \cong \Omega_{n-1,1}$ and $n \geq 5$ the first part follows from Proposition 8. Similarly, since $L_i \cap L_j \cong \Omega_{n-2,2}$, again by Proposition 8 the complex $L_i \cap L_j$ is connected if $n \geq 5$. \square

Theorem 10. *The complex Ω_n is μ_n -connected, where $\mu_n = \lfloor \frac{2n-1}{3} \rfloor - 2$.*

Proof: For $n = 2$ the complex Ω_2 consists of two points and is nonempty or (-1) -connected. For $n = 3$ the complex Ω_3 is also nonempty ((-1) -connected), being an union of two disjoint circles. The complex Ω_4 is 0-connected. Indeed, each pair v_0, v_1 of vertices in Ω_4 belongs to a subcomplex isomorphic to $\Omega_{2,2}$ which is connected.

Let us assume that $n \geq 5$. We already know that $\pi_1(\Omega_n) = 0$ so it remains to be shown that $H_j(\Omega_n) \cong 0$ for $j \leq \mu_n$. We establish this fact by induction on n .

Let us suppose that the statement of the theorem is true for complexes Ω_{n-2} and Ω_{n-1} . Consider the subcomplex Θ_n of Ω_n formed by simplices having possibly a vertex of the type $(1, i)$ or $(j, 1)$ but not both. Here is an excerpt from the long homology exact sequence of the pair (Ω_n, Θ_n) .

$$\cdots \rightarrow H_{\mu_n}(\Theta_n) \rightarrow H_{\mu_n}(\Omega_n) \rightarrow H_{\mu_n}(\Omega_n, \Theta_n) \rightarrow \cdots \quad (4)$$

We need yet another exact sequence involving complexes Ω_n and Θ_n . For motivation, the reader is referred to [20] where similar sequences are constructed in the context of usual chessboard complexes.

Let us denote by Θ_n^1 the subcomplex of Θ_n consisting of simplices having one vertex of the type $(1, i)$ and all of their faces. Similarly, let Θ_n^2 be the subcomplex of Θ_n consisting of simplices having one vertex of the type $(j, 1)$ and all of their faces. We use the Mayer-Vietoris sequence for the decomposition $\Theta_n = \Theta_n^1 \cup \Theta_n^2$. Obviously $\Theta_n^1 \cap \Theta_n^2 = \Omega_{n-1}$ so we obtain the following exact sequence

$$\cdots \rightarrow H_{\mu_n}(\Theta_n^1) \oplus H_{\mu_n}(\Theta_n^2) \rightarrow H_{\mu_n}(\Theta_n) \rightarrow H_{\mu_n-1}(\Omega_{n-1}) \rightarrow \cdots \quad (5)$$

Since both Θ_n^1 and Θ_n^2 are complexes of type $\Omega_{n-1,1}$, they are $\mu_{n-1,1}$ -connected by Proposition 8. Since $\mu_{n-1,1} = \lfloor \frac{2n-1}{3} \rfloor - 2 = \mu_n$ we observe that $H_{\mu_n}(\Theta_n^1) \oplus H_{\mu_n}(\Theta_n^2) = 0$.

The complex Ω_{n-1} is μ_{n-1} -connected by the induction hypothesis, and $\mu_{n-1} = \lfloor \frac{2n-3}{3} \rfloor - 2 \geq \lfloor \frac{2n-1}{3} \rfloor - 2 - 1 = \mu_n - 1$. Therefore, $H_{\mu_n-1}(\Omega_{n-1}) = 0$.

These facts, together with the exactness of the sequence (5), allow us to conclude that $H_{\mu_n}(\Theta_n) = 0$.

The homology of the pair (Ω_n, Θ_n) is isomorphic to the homology of the quotient Ω_n/Θ_n . If we denote by $I_{i,j}$ (for $i \neq j$) the 1-simplex with endpoints $(1, i)$ and $(j, 1)$, the argument similar to the one from Proposition 2 shows that this quotient is homotopy equivalent to the wedge

$$\bigvee_{1 < i \neq j \leq n} (I_{i,j} * \Omega_{n-2}) / (\partial I_{i,j} * \Omega_{n-2}).$$

Each quotient $(I_{i,j} * \Omega_{n-2}) / (\partial I_{i,j} * \Omega_{n-2})$ is homotopy equivalent to a wedge of double suspensions of the complex Ω_{n-2} . These double suspensions are by the induction hypothesis $(\mu_{n-2} + 2)$ -connected, and $(\mu_{n-2} + 2) = \lfloor \frac{2n-5}{3} \rfloor \geq \lfloor \frac{2n-1}{3} \rfloor - 2 = \mu_n$. Therefore $H_{\mu_n}(\Omega_n, \Theta_n) = 0$.

Finally, from the exact sequence (4) we deduce $H_{\mu_n}(\Omega_n) = 0$, which completes our inductive argument. \square

Substituting $n = p + 1$ and taking the suspension, one immediately obtains the desired estimate for the connectivity of the complex $\text{Sym}_*^{(p)}$ introduced by Ault and Fiedorowicz in [2].

Corollary 11. *The complex $\text{Sym}_*^{(p)}$ is $\lceil \frac{2}{3}(p-1) \rceil$ -connected.*

Proof: Since by definition $\text{Sym}_*^{(p)} = \Sigma\Omega_{p+1}$, it follows that this complex is γ_p -connected where

$$\gamma_p = \left\lceil \frac{2(p+1)-1}{3} \right\rceil - 2 + 1 = \left\lceil \frac{2}{3}(p-1) \right\rceil.$$

□

8 Tightness of the bound

Our objective in this section is to explore how far from being tight is the connectivity bound established in Theorem 10. Our central result is Theorem 13 which states that the constant μ_n is the best possible at least if $n = 3k + 2$ for some $k \geq 1$.

8.1 The case $n = 3k + 2$

Proposition 12. *The inclusion $\Omega_5 \hookrightarrow \Delta_5$ induces an epimorphism*

$$H_2(\Omega_5) \xrightarrow{\alpha} H_2(\Delta_5) \cong \mathbb{Z}_3.$$

Moreover, for a class $[S]$ such that $\alpha([S])$ is a generator in $H_2(\Delta_5)$, one can choose the fundamental class of the 2-sphere $S \cong S^1 * S^0 \cong \Omega_{2,2} * \Delta_{2,1} \subset \Omega_5$, depicted in Figure 4 (a), where $\Omega_{2,2} \subset \Delta_{[2],[4]}$ and $\Delta_{2,1} \cong \Delta(\{(3, 5), (4, 5)\})$.

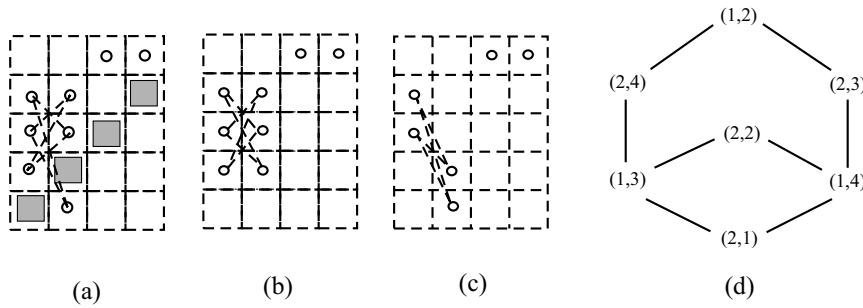


Figure 4: A non-zero class in $H_2(\Omega_5)$.

Proof: It is well known that $H_2(\Delta_5) \cong \mathbb{Z}_3$. Moreover, see [20] (Theorem 1.7. (i) and Lemma 5.9.) or [15] (Theorem 3.1.), a generator of this group is the fundamental class of the sphere $S_a^0 * S_b^0 * S_c^0$ where $S_a^0 = \Delta(\{(1, 1), (2, 1)\})$, $S_b^0 = \Delta(\{(3, 2), (3, 3)\})$ and $S_c^0 = \Delta(\{(4, 4), (5, 4)\})$. By permuting rows and columns of the chessboard $[5] \times [5]$ one

obtains other generators, in particular the fundamental class $[S_1]$ of the 2-sphere S_1 depicted in Figure 4 (c) is a non-trivial element in $H_2(\Delta_5)$. Let $[S_2]$ be the fundamental class of the 2-sphere S_2 shown in Figure 4 (b).

By assumption $S \cong S^1 * S^0$ is the 2-sphere shown in Figure 4 (a) where S^1 is the hexagon shown in Figure 4 (d) and $S^0 = \{(3, 5), (4, 5)\}$. The sphere S is a subcomplex of Ω_5 so it suffices to show that it defines a non-trivial element $[S]$ in the 2-homology of the larger complex Δ_5 . By inspection of Figure 4 (d) we observe that $[S] = \pm[S_1] \pm [S_2]$. The sphere S_2 bounds in Δ_5 since it is a subcomplex of the link of the vertex $(5, 1)$. From here it immediately follows that $[S] = \pm[S_1] \neq 0$. \square

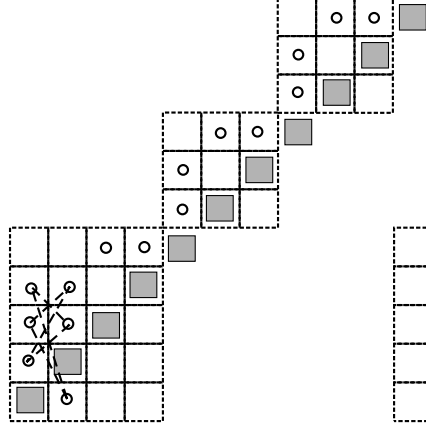


Figure 5: The complex Ω_{11} is not 6-connected.

Theorem 13. *The inclusion map $\Omega_{3k+2} \hookrightarrow \Delta_{3k+2}$ induces a non-trivial homomorphism*

$$H_{2k}(\Omega_{3k+2}) \longrightarrow H_{2k}(\Delta_{3k+2}).$$

It follows that $H_{2k}(\Omega_{3k+2})$ is non-trivial, hence the cycle-free chessboard complex Ω_{3k+2} is $(2k - 1)$ -connected but not $(2k)$ -connected for each $k \geq 1$.

Proof: We already know that the result is true in the case $k = 1$. The general case is not much more difficult to prove in light of the properties of chessboard complexes of the form Δ_{3k+2} established in [20]. For example Theorem 5.4. (loc. cit.) implies that $H_{2k}(\Delta_{3k+2}) \cong \mathbb{Z}_3$. Moreover, a generator of this group is determined by a sphere $S^{2k} \cong S^0 * \dots * S^0$ obtained as a join of $(2k + 1)$ copies of S^0 such that $(k + 1)$ of them are vertical and the remaining k are horizontal “dominoes”, i.e. complexes of the form $\Delta_{2,1}$ and $\Delta_{1,2}$ respectively. It is often convenient to represent two dominoes of different type inside a chessboard complex of the type $\Delta_{3,3}$, two of these (3×3) -chessboards with pairs of complementary dominos are indicated in Figure 5.

Let us illustrate the argument leading to the proof of the theorem in the case of the complex Ω_{11} , the proof of the general case follows exactly the same pattern. Figure 5 exhibits a sphere $\Sigma := S * S^1 * S^1 \cong S^6$, where S is the 2-sphere described in Proposition 12 while the two copies of S^1 arise from the dominos in two (3×3) -blocks.

It is clear that $\Sigma \subset \Omega_{11}$ so it remains to be shown that the image of Σ in Δ_{11} defines a non-zero homology class.

The image $\nu([S])$ of the class $[S] \in H_2(\Omega_5)$ in $H_2(\Delta_5)$ is shown in Proposition 12 to be non-trivial hence, according to Theorem 5.4. from [20], it is homologous (in Δ_5) to a sphere $S_1 = S^0 * S^0 * S^0$ where two of the “dominoes” S^0 are vertical. Hence $[\Sigma]$ is homologous (in Δ_{11}) to the fundamental class $[\Sigma_1]$ of $\Sigma_1 := S_1 * S^1 * S^1$ which, again by Theorem 5.4. from [20], is non-trivial. This completes the proof of the theorem. \square

8.2 The cases $n = 3k$ and $n = 3k + 1$

Unfortunately the methods used in this paper do not allow us to clarify if the constant μ_n from Theorem 10 is the best possible if $n = 3k$ or $n = 3k + 1$ for some $k \geq 1$. Nevertheless we are able to show that this bound should not be expected to be too far off the actual bound.

Proposition 14. *The group $H_{2k-1}(\Omega_{3k,1})$ is non-trivial. Moreover, a non-trivial element of this group arises as the fundamental class $\xi_{2k-1} = [\Sigma_{2k-1}]$ of a subcomplex $\Sigma_{2k-1} \subset \Omega_{3k,1}$ isomorphic to the join $S^0 * \dots * S^0 \cong S^{2k-1}$ of $2k$ copies of 0-dimensional spheres.*

Proof: Our model for $\Omega_{n,1} = \Omega_{3k,1}$ is the complex $\Omega_{X,Z}^{Y,0}$ where $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_n\}$, $Z = \{x_0\}$ and the bijection $\alpha : Y \rightarrow X$ maps y_j to x_j . The case $k = 3$ is depicted in Figure 6 where the shaded squares correspond (from left to right) to squares (x_j, y_j) while the first column is filled with squares (x_0, y_j) , $j = 1, \dots, n$.

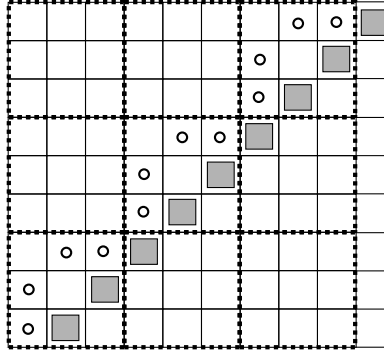


Figure 6: A cycle in Δ_9 .

Let $T_1 := \{(x_0, y_1), (x_0, y_2)\}$, $T_2 := \{(x_1, y_3), (x_2, y_3)\}$, $T_3 := \{(x_3, y_4), (x_3, y_5)\}$, \dots , $T_{2k-1} := \{(x_{3k-3}, y_{3k-2}), (x_{3k-3}, y_{3k-1})\}$, $T_{2k} := \{(x_{3k-2}, y_{3k}), (x_{3k-1}, y_{3k})\}$. Define Σ_{2k-1} as the join $T_1 * \dots * T_{2k}$. The proof is completed by the observation that the cycle ξ_{2k-1} , determined by the sphere Σ_{2k-1} , does not bound even in the larger chessboard complex $\Delta_{X \cup \{x_0\}, Y}$, cf. [20], Section 3. \square

The following corollary provides evidence that the connectivity bound established in Theorem 10 is either tight or very close to the actual connectivity bound in the two remaining cases, $n = 3k, n = 3k + 1$.

Corollary 15. For each $k \geq 1$,

$$\text{either } H_{2k-1}(\Omega_{3k}) \neq 0 \quad \text{or} \quad H_{2k-1}(\Omega_{3k+1}) \neq 0.$$

Proof: Let Ω_{3k+1} be the cycle-free chessboard complex based on the chessboard $[3k+1] \times [3k+1]$. Define $\Omega_{3k,1}$ as the subcomplex of Ω_{3k+1} such that a simplex $S \in \Omega_{3k+1}$ is in $\Omega_{3k,1}$ if and only if $S \cap (\{1\} \times [3k+1]) = \emptyset$. The quotient complex $\Omega_{3k+1}/\Omega_{3k,1}$ has the homotopy type of a wedge $\bigvee_{i=1}^{3k+1} \Sigma(\Omega_{3k}^{(i)})$ where each of the complexes $\Omega_{3k}^{(i)}$ is isomorphic to Ω_{3k} . Consider the following fragment of the long exact sequence of the pair $(\Omega_{3k+1}, \Omega_{3k,1})$,

$$\dots \rightarrow \bigoplus_{i=1}^{3k+1} H_{2k-1}(\Omega_{3k}^{(i)}) \rightarrow H_{2k-1}(\Omega_{3k,1}) \rightarrow H_{2k-1}(\Omega_{3k+1}) \rightarrow \dots$$

The desired conclusion follows from the fact that $H_{2k-1}(\Omega_{3k,1}) \neq 0$. \square

Conjecture: The connectivity bound given in Theorem 10 is the best possible or in other words for each $n \geq 2$,

$$H_{\mu_n+1}(\Omega_n) \neq 0.$$

9 Relatives of Ω_n

The closest relative of Ω_n , that has so far appeared in the literature, is the complex Δ_n^{DM} of directed matchings introduced by Björner and Welker in [6], see also Section 2.1. In this section we describe a natural “ecological niche” for all these complexes and briefly compare their connectivity properties.

In the sequel we put more emphasis on the directed graph description of Ω_n, Δ_n and related complexes (Section 2.1). We silently identify a directed graph with its set of directed edges (assuming the set of vertices is fixed and clear from the context).

Let DK_n be the complete directed graph on the set $[n]$ of vertices (directed loops included) and K_n^\uparrow its companion with all loops excluded.

Following [6], let Δ_n^{DM} be the directed graph complex of all *directed matchings* in K_n^\uparrow . By definition, $\Gamma \subset K_n^\uparrow$ is a directed matching if both the in-degree and out-degree of each vertex is at most one. This is equivalent to the condition that the two graphs depicted in Figure 2 (a) are banned from Γ . It follows that $\Gamma \subset DK_n$ is in Δ_n^{DM} if and only if the connected components of Γ are either directed paths or directed cycles of length at least 2, i.e. the only difference between Δ_n and Δ_n^{DM} is that in the former complex the cycles of length one (loops) are allowed.

Summarizing, if $\Gamma \subset DK_n$ then

- (1) $\Gamma \in \Delta_n \Leftrightarrow$ Each connected component of Γ is either a directed path or a directed cycle,
- (2) $\Gamma \in \Delta_n^{DM} \Leftrightarrow$ Each connected component of Γ is either a directed path or a directed cycle of length at least 2,

(3) $\Gamma \in \Omega_n \Leftrightarrow$ Each connected component of Γ is a directed path.

The following definition introduces (some of) natural intermediate complexes which interpolate between Ω_n and Δ_n , respectively Ω_n and Δ_n^{DM} .

Definition 16. Let $F_p^I = F_p(\Delta_n)$, respectively $F_p^{II} = F_p(\Delta_n^{DM})$, be the subcomplex of Δ_n (respectively Δ_n^{DM}) such that $\Gamma \in F_p(\Delta_n)$ (respectively $\Gamma \in F_p(\Delta_n^{DM})$) if and only if the number of cycles, among the connected components in Γ is at most p .

Definition 17. A graph $C \subset DK_n$ is a p -multicycle if $C = C_1 \uplus C_2 \uplus \dots \uplus C_p$ has exactly p connected components and each C_j is a cycle. If $l_j := l(C_j)$ is the length of C_j then the multiset $t(C) := (l_1, l_2, \dots, l_p)$ is called the type of C and the number $l(C) := l(C_1) + \dots + l(C_k)$ is called the length of the p -multicycle C . Let \mathcal{C}_p be the set of all p -multicycles and $\mathcal{C}_p^{\geq 2}$ the set of all p -multicycles C of type $t(C) := (l_1, l_2, \dots, l_p)$ such that $l_j \geq 2$ for each j .

Proposition 18.

$$F_0^I = F_0^{II} = F_0(\Delta_n) = F_0(\Delta_n^{DM}) = \Omega_n \quad (6)$$

$$F_p^I / F_{p-1}^I \simeq \bigvee_{C \in \mathcal{C}_p} S^{l(C)-1} * \Omega_{n-l(C)} \cong \bigvee_{C \in \mathcal{C}_p} \Sigma^{l(C)}(\Omega_{n-l(C)}) \quad (7)$$

$$F_p^{II} / F_{p-1}^{II} \simeq \bigvee_{C \in \mathcal{C}_p^{\geq 2}} S^{l(C)-1} * \Omega_{n-l(C)} \cong \bigvee_{C \in \mathcal{C}_p^{\geq 2}} \Sigma^{l(C)}(\Omega_{n-l(C)}) \quad (8)$$

The associated exact (spectral) sequences show that all these complexes are closely related, in particular have very similar connectivity properties. Let $\mu_n = \lfloor \frac{2n-1}{3} \rfloor - 2$ and $\nu_n = \lfloor \frac{2n+1}{3} \rfloor - 2$. The complex Δ_n is ν_n -connected, as demonstrated by Björner et al. in [5]. The same connectivity bound was established by Björner and Welker for Δ_n^{DM} in [6]. Both bounds are tight as proved by Sharesian and Wachs in [20]. It follows from Proposition 18 that the majority of complexes F_p^I and F_p^{II} share this connectivity bound. On the other hand Ω_n is by Theorem 10 μ_n -connected, hence all these complexes are $\mu_n = \nu_n$ connected if $n = 3k + 2$ for some k .

References

- [1] C.A. Athanasiadis. Decompositions and connectivity of matching and chessboard complexes. *Discrete Comput. Geom.* 31 (2004), 395–403.
- [2] S. Ault, Z. Fiedorowicz. Symmetric homology of algebras. arXiv:0708.1575v54 [math.AT] 5 Nov 2007.
- [3] E. Babson, A. Björner, S. Linusson, J. Sharesian, and V. Welker. Complexes of not i -connected graphs. *Topology*, 38(2):271–299, 1999.
- [4] A. Björner. *Topological methods*, in *Handbook of Combinatorics* (R. Graham, M. Grötschel, and L. Lovász, eds.), North-Holland, Amsterdam, 1995, 1819–1872.

- [5] A. Björner, L. Lovász, S.T. Vrećica, and R.T. Živaljević. Chessboard complexes and matching complexes. *J. London Math. Soc. (2)* 49 (1994), 25–39.
- [6] A. Björner, V. Welker. Complexes of directed graphs. *SIAM J. Discrete Math.*, 12(4):413–424, 1999.
- [7] S. Bouc. Homologie de certains ensembles de 2-sous-groupes des groupes symétriques. *J. Algebra* 150 (1992), 158–186.
- [8] R. Brown. *Elements of modern topology*. McGraw-Hill, London 1968.
- [9] Z. Fiedorowicz. Question about a simplicial complex. *Algebraic Topology Discussion List* (maintained by Don Davis), <http://www.lehigh.edu/~dmd1/zf93>.
- [10] J. Friedman and P. Hanlon. On the Betti numbers of chessboard complexes. *J. Algebraic Combin.* 8 (1998), 193–203.
- [11] P.F. Garst. *Cohen-Macaulay complexes and group actions*. Ph.D. Thesis, Univ. of Wisconsin-Madison, 1979.
- [12] J. Jonsson. *Simplicial Complexes of Graphs*, Ph.D thesis, KTH Stockholm 2005. <http://urn.kb.se/resolve?urn=urn:nbn:se:kth:diva-202>.
- [13] J. Jonsson. *Simplicial Complexes of Graphs*. Lecture Notes in Mathematics , Vol. 1928. Springer 2008.
- [14] J. Jonsson. Exact sequences for the homology of the matching complex. Preprint, 2007.
- [15] J. Jonsson. On the 3-torsion part of the homology of the chessboard complex. Preprint, 2007.
- [16] J. Leray. Sur la forme des espaces topologiques et sur les points fixes des représentations. *J. Math. Pures Appl.* 24 (1945), 95–167.
- [17] J-L. Loday. *Cyclic homology*. Springer-Verlag, Berlin, 1998.
- [18] V. Reiner and J. Roberts. Minimal resolutions and homology of chessboard and matching complexes. *J. Algebraic Combin.* 11 (2000), 135–154.
- [19] K.S. Sarkaria. A generalized van Kampen-Flores theorem. *Proc. Amer. Math. Soc.* 111 (1991), 559–565.
- [20] J. Shareshian and M.L. Wachs. Torsion in the matching complex and chessboard complex. *Advances in Mathematics* 212 (2007) 525–570.
- [21] V. Turchin. Homologies of complexes of doubly connected graphs. *Russian Math. Surveys (Uspekhi)*, 52:426–427, 1997.

- [22] V.A. Vassiliev. Topology of two-connected graphs and homology of spaces of knots. In S. Tabachnikov, editor, *Differential and Symplectic Topology of Knots and Curves (Amer. Math. Soc. Transl. Ser. 2, vol. 98)*, pages 253–286. Amer. Math. Soc., 1999.
- [23] M.L. Wachs. Topology of matching, chessboard, and general bounded degree graph complexes, Dedicated to the memory of Gian-Carlo Rota. *Algebra Universalis* 49 (2003), 345–385.
- [24] G.M. Ziegler. Shellability of chessboard complexes. *Israel J. Math.* 87 (1994), 97–110.
- [25] R.T. Živaljević and S.T. Vrećica. The colored Tverberg’s problem and complexes of injective functions. *J. Combin. Theory Ser. A* 61 (1992), 309–318.