

Chessboard complexes indomitable*

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Abstract

We give an alternative proof of the striking new Tverberg type theorem of Blagojević and Ziegler [B-Z]. Our method also yields some new cases of “constrained Tverberg theorem” in the sense of Hell [H], including a simple colored Radon’s theorem for $d + 3$ points in \mathbb{R}^d (Corollary 7). This gives us an opportunity to review some of the highlights and reexamine the role of chessboard complexes in these and related problems of topological combinatorics.

1 Introduction

Chessboard (simplicial) complexes and their relatives have been for decades an important theme of topological combinatorics with often unexpected applications in group theory, representation theory, commutative algebra, Lie theory, computational geometry, algebraic topology, and combinatorics, see [Ata], [A-F], [BLVZ], [F-H], [G], [J1], [S-W], [VŽ94], [VŽ09], [W], [Z], [ŽV92]. The books [J] and [M], as well as the review papers [W] and [Ž04], cover selected topics of the theory of chessboard complexes and contain a more complete list of related publications.

Chessboard complexes originally appeared in [G] as coset complexes of the symmetric group, closely related to Coxeter and Tits coset complexes. In combinatorics they appeared as “complexes of partial injective functions” [ŽV92], “multiple deleted joins” [ŽV92], complexes of all partial matchings in complete bipartite graphs, and the complexes of all non-taking rook configurations [BLVZ].

Recently a naturally defined subcomplex of the chessboard complex, referred to as the “cycle-free chessboard complex”, has emerged in the context of stable homotopy theory ([A-F] and [Fie]), where it was introduced as a tool for evaluating the symmetric group analogue for the cyclic homology of algebras.

*This is an expanded version, with added Proposition 4 and its consequences, of our note [X].

In our own research [ŽV92, VŽ94] chessboard complexes appeared as a tool for the resolution of the well known *colored Tverberg problem*, see [M] and [Ž04] for background details including the connections with other well known problems of discrete and computational geometry. In these papers the role of chessboard complexes for Tverberg type problems was discovered, and the importance of Borsuk-Ulam type questions for equivariant maps defined on joins of chessboard complexes recognized.

Next fifteen years witnessed little progress and it is probably fair to say that majority of specialists, including ourselves, grew to believe that the limits of the method are reached and a new progress towards better bounds in the colored Tverberg problem difficult to expect.

Consequently it was indeed a wonderful surprise when Pavle Blagojević and Günter Ziegler [B-Z] proved the opposite and established so far the most natural and elegant version of (type A) colored Tverberg theorem.

Motivated by the breakthrough of Blagojević and Ziegler we prove some new Borsuk-Ulam type results for joins of chessboard complexes (Propositions 2, 3 and 4) leading to a shorter and conceptually simpler proof¹ of their main result. Among other consequences of our approach are new constrained Tverberg theorems close in spirit to results of S. Hell [H], the simplest example being the “Colored Radon’s theorem” (Corollary 7).

2 Preliminaries

2.1 Chessboard complexes $\Delta_{m,n}$

A function $f : A \rightarrow B$ can be interpreted as a *coloring* of elements of a set A by colors from a set B . A partial coloring of A is a function $\phi : D \rightarrow B$ where $D = D(\phi)$ the domain of ϕ is a subset of A . By convention a partial function ϕ from A to B is often identified with its graph $\Gamma(\phi) = \{(i, \phi(i)) \mid i \in D(\phi)\} \subset A \times B$. It follows that if A and B are finite then the set $\text{Col}(A, B)$ of all (partial) colorings of A by colors from B is a simplicial complex on $A \times B$ as a set of vertices. If the cardinalities of sets A and B are respectively m and n then it is immediate that the complex $\text{Col}(A, B)$ is isomorphic to the join $A^{*n} \cong [m]^{*n}$ of n copies of an m -element set (0-dimensional complex) A .

If we restrict our attention to partial injective functions ϕ we immediately arrive at the chessboard complex

$$\Delta_{m,n} := \Delta_{A,B} = \{\Gamma(\phi) \in \text{Col}(A, B) \mid \phi \text{ is an injective function}\}$$

in the form it was introduced in [ŽV92, VŽ94] in the context of the colored Tverberg problem. The name “chessboard complex” comes from the fact that each simplex

¹After the preliminary version [X] of our paper was released and shared with a circle of specialists, we were kindly informed by P. Blagojević that B. Matschke has also discovered a proof that simplifies their original approach.

$\Gamma(\phi)$ can be interpreted as a non-taking rook placement on a $(m \times n)$ -chessboard $A \times B \cong [m] \times [n]$.

The “injectivity restriction” on a coloring is quite natural from the point of view of “constrained colorings” in the sense of Hell [H], when only some of the colorings are admissible. For example if A is the vertex set of a graph $\Gamma = (A, E)$, then it is natural to ask that ϕ is a proper coloring in the sense that two adjacent vertices are always colored by different colors.

If $A = [m], B = [n]$ and $\Gamma = K_m$ is the complete graph, then the simplicial complex of all admissible (partial) colorings of A by n distinct colors is precisely the chessboard complex $\Delta_{m,n}$. More generally if Γ is a disjoint union of cliques, i.e. if there is a partition $A = A_1 \cup \dots \cup A_k$ such that $\Gamma = K_{A_1} \cup \dots \cup K_{A_k}$ is the union of complete graphs, then the complex of admissible colorings is isomorphic to the join

$$\Delta_{A_1,B} * \dots * \Delta_{A_k,B} \tag{1}$$

of chessboard complexes.

2.2 Colored Tverberg problem

Here we very briefly review how the joins of chessboard complexes naturally arise in the *Colored Tverberg problem* and related problems of geometric combinatorics.

Suppose that $C \subset \mathbb{R}^d$ is a finite set and let $\psi : C \rightarrow [k + 1]$ be a coloring of this set (Section 2.1) by $(k + 1)$ colors where $k \leq d$. The coloring is always supposed to be a *strict coloring* in the sense that the coloring function ψ is an epimorphism, i.e. that all listed colors are used.

A subset $X \subset C$ is called *multicolored* if the restriction of ψ on X is injective, i.e. if elements of X are all colored by different colors. In this case the (possibly degenerated) simplex $\sigma := \text{conv}(X)$ is also referred to as a multicolored or a *rainbow* simplex.

- The Colored Tverberg problem asks in general for conditions on the coloring function ψ which guarantee the existence of many, vertex-disjoint rainbow simplices which have a non-empty intersection.

If the colors are evenly distributed we are led to the following version of the general problem in the form it was recorded in [Ž04].

Problem 1. *For given integers r, k, d such that $k \leq d$, determine the minimum number $t = t(r, k, d)$ such that for any $C \subset \mathbb{R}^d$ of size $t(k + 1)$ and each strict coloring $\psi : C \rightarrow [k + 1]$ such that $C_j := \psi^{-1}(j)$ has t elements, there exist r disjoint, multicolored sets $B_j \subset C$ such that*

$$\bigcap_{j=1}^r \text{conv}(B_j) \neq \emptyset. \tag{2}$$

The reader is again referred to the book [M] and reviews [Ž04, Ž98] for a much more detailed presentation with a fairly complete set of references. Here we recall only that the problem of evaluating $t(r, d, d)$ was originally proposed by Bárány and Larman [BL], after it was observed by Bárány, Füredi and Lovász [BFL] that the “weak colored Tverberg theorem” $t(r, d, d) < +\infty$ resolves a number of interesting conjectures in discrete and computational geometry (halving hyperplanes problem, point selection problem, weak ϵ -net problem, hitting set problem).

After the preliminary result [BL] that $t(r, 2, 2) = r$ and $t(2, d, d) = 2$, the general bounds $t(r, d, d) \leq 2r - 1$ and $t(r, k, d) \leq 2r - 1$ were established in [ŽV92], respectively [VŽ94] for all primes r . Subsequently [Ž98], and without introducing really new ideas, the result was extended to the case of prime powers. Note that the distinction between the cases $k = d$ and $k < d$ is important since in the latter case (for dimensional reasons) there is an additional constraint $r \leq d/(d - k)$. For this reason the case $k = d$ is often referred to as the “type A colored Tverberg problem” while the case $k < d$ is known as the “type B colored Tverberg problem”, see [Ž04].

Note that the bound $t(r, k, d) \leq 2r - 1$ for $k < d$ was shown in [VŽ94] to be tight, while the central new result of [B-Z] (confirming the conjecture from [BL]) is the equality $t(r, d, d) = r$ for prime r .

2.3 Topology enters the scene

The essence of the original breakthrough [ŽV92, VŽ94], leading to the inequality $t(r, k, d) \leq 2r - 1$, was the observation that the colored Tverberg problem is closely related to a question of Borsuk-Ulam type for joins of chessboard complexes. More precisely it was shown that both type A and type B cases of the problem follow from the non-existence of a \mathbb{Z}/r -equivariant map

$$(\Delta_{r, 2r-1})^{*(k+1)} \longrightarrow S(W_r^{\oplus d}) \quad (3)$$

where W_r is the real, $(r - 1)$ -dimensional permutation representation of the cyclic group \mathbb{Z}/r .

Indeed, let $C \subset \mathbb{R}^d$ be a set of size $(2r - 1)(k + 1)$ and let $\psi : C \rightarrow [k + 1]$ be a coloring function such that $C_j := \psi^{-1}(j)$ has size $(2r - 1)$. We are supposed to show that there exist pairwise disjoint subsets B_j , $j = 1, \dots, k + 1$, satisfying (2) such that ψ is injective on them. In light of (1) it is plausible and not at all difficult to see that this is indeed a consequence of (3).

2.4 The breakthrough of Blagojević and Ziegler

It is amusing to see how ingenious and astonishingly simple was the new idea of Blagojević and Ziegler [B-Z] leading to the bound $t(r - 1, d, d) = r - 1$ for a prime r . They observed that if $C \subset \mathbb{R}^d$ is a set of size $(r - 1)(d + 1)$ which is evenly colored by $(r - 1)$ colors, then it is natural to add one more point $x \in \mathbb{R}^d$ and one more color (which corresponds to the added point x). The enlarged set $C^+ = C \cup \{x\}$ is colored by r colors and it is natural to ask whether one can find r vertex-disjoint rainbow simplices

which have a non-empty intersection. Here, as in Section 2.1, a simplex is rainbow if all its vertices are colored by different colors.

By using exactly the same translation as above, and in perfect analogy with (3), one is immediately led to the question of the existence of a \mathbb{Z}/r -equivariant map

$$F : (\Delta_{r,r-1})^{*d} * [r] \rightarrow S(W_r^{\oplus d}).$$

3 Main results

Proposition 2. *The degree $\deg(f)$ of each \mathbb{Z}/r -equivariant map*

$$f : (\Delta_{r,r-1})^{*d} \rightarrow S(W_r^{\oplus d}) \quad (4)$$

*is non-zero, provided r is a prime number. Moreover $\deg(f) \equiv_{\text{mod } r} (-1)^d$ and for any integer m such that $m \equiv_{\text{mod } r} (-1)^d$ there exists a \mathbb{Z}/r -equivariant map $g : (\Delta_{r,r-1})^{*d} \rightarrow S(W_r^{\oplus d})$ such that $\deg(g) = m$.*

Proposition 2 implies the following proposition which has the key technical Proposition 4.2. from [B-Z] as an immediate consequence.

Proposition 3. *Let $r \geq 2$ be a prime and $d \geq 1$. Then there does not exist a \mathbb{Z}/r -equivariant map*

$$F : (\Delta_{r,r-1})^{*d} * [r] \rightarrow S(W_r^{\oplus d}). \quad (5)$$

Proposition 3 establishes the main colored Tverberg type result of [B-Z]. However, the formal similarity of statements (3), (4), and (5), and our general emphasis on the equivariant maps from joins of chessboard complexes, motivate us to pursue this idea further.

Proposition 4. *Suppose that X is a $(\nu - 1)$ -connected, free \mathbb{Z}/r -complex where r is a prime number. Suppose that*

$$U \cong W_r^{\oplus l} \oplus V$$

where W_r and V are two fixed point free representation of \mathbb{Z}/r , V of dimension $\leq \nu$ and W_r the $(r - 1)$ -dimensional permutation representation of \mathbb{Z}/r . Then there does not exist a \mathbb{Z}/r -equivariant map

$$f : (\Delta_{r,r-1})^{*l} * X \rightarrow S(W_r^{\oplus l} \oplus V). \quad (6)$$

Using the known fact [BLVZ] that $\Delta_{s,t} \cong \Delta_{s,t}$ is $(\nu - 1)$ -connected where $\nu = \min\{s, t, \lfloor \frac{1}{3}(s+t+1) \rfloor\} - 1$, Proposition 4 specializes to results claiming non-existence of equivariant maps of the form

$$f : (\Delta_{r,r-1})^{*l} * \Delta_{r,s_1} \dots * \Delta_{r,s_k} \rightarrow S(W_r^{\oplus(d+1)}) \quad (7)$$

for an appropriate choice of parameters s_1, \dots, s_k and l allowing an application of Proposition 4. Since $[r]$ is nothing but $\Delta_{r,1}$ we observe that Proposition 3 is the simplest instance of (7).

The case $s_1 = \dots = s_l$ is of special interest and all this together implies that there should exist a plethora of colored Tverberg results of mixed type A and type B in the sense of [Z04].

4 Colored Tverberg results of mixed type

Here we specialize further and list the first consequences of Proposition 4. We initially focus our attention to the case $s_1 = \dots = s_k$.

4.1 The case $s_1 = \dots = s_k = 2r - 1$

Since $\Delta_{r,2r-1}$ is $(r-2)$ -connected the complex $\Delta_{r,2r-1}^{*k}$ is $(rk-2)$ -connected. It follows from Proposition 4 that there does not exist a \mathbb{Z}/r -equivariant map

$$f : (\Delta_{r,r-1})^{*l} * (\Delta_{r,2r-1})^{*k} \rightarrow S(W_r^{\oplus(d+1)}) \cong S(W_r^{\oplus l}) * S(W_r^{\oplus(d-l+1)}) \quad (8)$$

provided

$$(r-1)(d-l+1) + 1 \leq rk. \quad (9)$$

From here one immediately deduces the following proposition.

Proposition 5. *Suppose that $C \subset \mathbb{R}^d$ is a collection of $N = (r-1)l + (2r-1)k$ points in \mathbb{R}^d colored by $k+l$ colors and let $C = \cup_{j=1}^{k+l} C_j$ be the associated partition of C into monochromatic parts. Assume that $|C_i| = r-1$ for $i = 1, \dots, l$ and $|C_i| = 2r-1$ for $i = l+1, \dots, l+k$. Assume that the inequality (9) is satisfied and that r is a prime number. Then there exist r vertex disjoint rainbow simplices which have a non-empty intersection.*

In the case when $k = 1$ we obtain the following Corollary.

Corollary 6. *Suppose that C_1, \dots, C_d, C_{d+1} are (colored) sets in \mathbb{R}^d such that C_{d+1} has $2r-1$ elements while each of the remaining sets has cardinality $r-1$. Then one can find r vertex-disjoint rainbow simplices with a non-empty intersection.*

Let us compare Corollary 6 to the original result of Blagojević and Ziegler [B-Z]. For prime r they ask for $r-1$ intersecting rainbow simplices and we ask for r . If we neglect for a moment a necessary assumption for these numbers to be prime, and if we ask for r intersecting rainbow simplices in both cases, [B-Z] requires r points in each of $d+1$ colors whereas we ask for $r-1$ points of d colors and $2r-1$ points of the last color. The difference of the total numbers of points in both cases is $(r-1)d+2r-1-r(d+1) = r-(d+1)$. So, in some (not direct) sense, their result gives more when r is greater than $d+1$, and our in the other case.

For illustration (the case $r = 2$) here is a colorful extension of Radon's theorem.

Corollary 7. *(Colored Radon theorem) Let C be a collection of $d+3$ points in \mathbb{R}^d , three of the same color and the remaining points all of different colors. Then there exist two vertex disjoint rainbow simplices which have a non-empty intersection.*

The following corollary shows that Proposition 5 is in some sense a mixed type A and type B colored Tverberg theorem, [Ž04].

Corollary 8. *If $l = 0$ then the inequality (9) reduces to $r \leq d/(d-k+1)$ and Proposition 5 reduces to the type B colored Tverberg theorem, [VŽ94, Ž04].*

4.2 The case $r = 2p - 1$ and $s_1 = \dots = s_k = p$

Suppose that $r = 2p - 1$ is an odd prime. It follows from Proposition 4 that there does not exist a \mathbb{Z}/r -equivariant map

$$f : (\Delta_{2p-1, 2p-2})^{*l} * (\Delta_{2p-1, p})^{*k} \rightarrow S(W_r^{\oplus(d+1)}) \cong S(W_r^{\oplus l}) * S(W_r^{\oplus(d-l+1)}) \quad (10)$$

provided

$$(r - 1)(d - l + 1) + 1 \leq pk. \quad (11)$$

Proposition 9. *Suppose that $C \subset \mathbb{R}^d$ is a collection of $N = (r - 1)l + pk$ points in \mathbb{R}^d colored by $k + l$ colors and let $C = \cup_{j=1}^{k+l} C_j$ be the associated partition of C into monochromatic parts. Assume that $|C_i| = r - 1$ for $i = 1, \dots, l$ and $|C_i| = p$ for $i = l + 1, \dots, l + k$. Assume that the inequality (11) is satisfied and that r is a prime number. Then there exist r vertex disjoint rainbow simplices which have a non-empty intersection.*

Proposition 9 specializes for particular values of parameters $r = 2p - 1, k, l, d$ to results that also deserve closer inspection.²

For example if $l = 0$ then the condition (11) is fulfilled if we assume the equality $pk = (r - 1)(d + 1) + 1$. Since on the right hand side is precisely the number of points needed for the classical (monochromatic) Tverberg theorem, we observe that Proposition 9 implies its direct colored refinements.

Example 10. *Choose $d = 4, p = 3, r = 5, k = 7$. Then Proposition 9 says that if 21 point in \mathbb{R}^4 is colored by 7 colors then there always exist 5 vertex-disjoint rainbow simplices with a non-empty intersection.*

5 Proofs

5.1 Degrees of equivariant maps

The proof of Proposition 2 relies on a general result (Proposition 11) which can be seen as a relative of theorems about mapping degrees of equivariant maps between representation spheres, see [tD] Proposition 4.12. (chapter II) for an example, and page 139 for a brief guide to other results of similar nature.

Proposition 11. *Suppose that M is a triangulated, compact, \mathbb{Z} -orientable, n -dimensional pseudo-manifold. Let G be a finite group which acts freely and simplicially on M and let $S(W)$ be a G -invariant sphere in a real, $(n + 1)$ -dimensional G -representation W . Suppose that each $g \in G$ either changes or preserves orientations of both M and $S(W)$. Then for any two G -equivariant maps $f, g : M \rightarrow S(W)$,*

$$\deg(f) \equiv \deg(g) \pmod{|G|}. \quad (12)$$

²A more complete discussion of these and other consequences of Proposition 4 will be given in a subsequent version of this paper.

Proof: Let $F : M \times I \rightarrow V$ be a G -equivariant homotopy between maps $i \circ f$ and $i \circ g$ transverse to $0 \in V$, where $i : S(V) \rightarrow V$ is the inclusion map. Since the subspace $\Sigma \subset M$ of singular points has dimension $\leq n - 2$, the set $\Sigma \times I$ has dimension $\leq n - 1$, hence we can assume that $0 \notin F(\Sigma \times I)$.

It follows that the set $Z(F) := F^{-1}(0)$ is finite and consists of non-singular points. The set $Z(F)$ is clearly G -invariant. For each $x \in Z(F)$ choose an open ball $V_x \ni x$ such that $V := \cup_{x \in Z(F)} V_x$ is G -invariant and $V_x \cap V_y \neq \emptyset$ for $x \neq y$. Let $S_x^n := \partial(V_x) \cong S^n$ be the boundary of V_x .

Let $N := (M \times I) \setminus V$, $M_0 := M \times \{0\}$ and $M_1 := M \times \{1\}$. By construction there is a relation among (properly oriented) fundamental classes,

$$[M_1] - [M_0] = \sum_{x \in Z(F)} [S_x^n] \quad (13)$$

inside the homology group $H_n(N, \mathbb{Z})$. The map $F_* : H_n(N, \mathbb{Z}) \rightarrow H_n(S(W), \mathbb{Z})$ maps the relation (13) into the desired congruence (12). \square

5.2 Canonical equivariant maps

Proposition 11 reduces the problem of evaluating the mod (r) degree of an arbitrary equivariant map $f : M \rightarrow S(V)$ to the much easier problem of testing a well chosen (canonical) map of this kind.

Definition 12. Let $[m]^{(k)}$ be the collection of all p -element subsets of $[r]$ for $p \leq k$, i.e. $[m]^{(k)}$ is the $(k - 1)$ -skeleton of an $(m - 1)$ -dimensional simplex. Define

$$\pi = \pi_{m,k} : \Delta_{m,k} \rightarrow [m]^{(k)} \quad (14)$$

as the projection which sends a non-taking rook placement $S = \{(i_1, j_1), \dots, (i_p, j_p)\} \subset [m] \times [k]$ to the set $\pi(S) = \{i_1, \dots, i_p\} \subset [m]$.

Proposition 13. The degree of the map $\pi_{r,r-1} : \Delta_{r,r-1} \rightarrow [r]^{(r-1)}$ is

$$\deg(\pi_{r,r-1}) = (r - 1)!$$

Proof: The proof follows from a simple algebraic count of points in the pre-image $\pi^{-1}(x)$ where x is the barycenter of a top dimensional simplex in $[r]^{(r-1)}$. \square

5.3 Proofs of Propositions 2, 3 and 4

Proof of Proposition 2: According to [BLVZ] the chessboard complex $\Delta_{r,r-1}$ is an orientable pseudo-manifold with a free action of the group \mathbb{Z}/r . The same holds for the join $(\Delta_{r,r-1})^{*d}$ so, in light of Proposition 11, it is sufficient to exhibit a canonical, \mathbb{Z}/r -equivariant map

$$\pi : (\Delta_{r,r-1})^{*d} \rightarrow S(W_r^{\oplus d})$$

with a known degree. Since $S(W_r) \cong [r]^{(r-1)}$ as \mathbb{Z}/r -spaces, and

$$S(W_r^{\oplus d}) \cong S(W_r)^{*d} \cong ([r]^{(r-1)})^{*d},$$

we observe that $\pi := (\pi_{r,r-1})^{*d}$ is an example of such a map. Hence the desired relation $\deg(\pi) = [(r-1)!]^d \equiv_{\text{mod } r} (-1)^d$ is a consequence of Proposition 13. \square

Proof of Proposition 3: Since the cone

$$\text{Cone}[(\Delta_{r,r-1})^{*d}] \cong (\Delta_{r,r-1})^{*d} * [1]$$

is a subcomplex of $(\Delta_{r,r-1})^{*d} * [r]$, we observe that if an equivariant map

$$F : (\Delta_{r,r-1})^{*d} * [r] \rightarrow S(W_r^{\oplus d})$$

exists, then its restriction on the subcomplex $(\Delta_{r,r-1})^{*d}$ would have a zero degree, which is in contradiction with Proposition 2. \square

Proof of Proposition 4: Suppose that there exists a \mathbb{Z}/r -equivariant map described in line (6). Since X is $(\nu-1)$ -connected and $S(V)$ is $(\nu-1)$ -dimensional, there exists a \mathbb{Z}/r -equivariant map $\alpha_1 : S(V) \rightarrow X$. Consequently there exists a \mathbb{Z}/r -equivariant map

$$\alpha = Id * \alpha_1 : (\Delta_{r,r-1})^{*d} * S(V) \rightarrow (\Delta_{r,r-1})^{*d} * X \quad (15)$$

and the composite map

$$f \circ \alpha : (\Delta_{r,r-1})^{*d} * S(V) \rightarrow S(W_r^{\oplus d} \oplus V) = S(W_r^{\oplus d}) * S(V). \quad (16)$$

It follows from Proposition 2 and Proposition 11 that the degree of this map is non-zero. On the other hand the degree of this map must be zero since the $f \circ \alpha$ -image of the fundamental class of $(\Delta_{r,r-1})^{*d} * S(V)$ must be zero, a consequence of the fact that the α_1 -image of $[S(V)]$ is zero in $H_*(X, \mathbb{Z})$. \square

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