Equipartitions of a Mass in Boxes

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Abstract

The aim of this paper is to provide the sufficient condition for a mass distribution in $\mathbb{R}^d$ to admit an equipartition with a collection of hyperplanes some of which are parallel. The results extend the previously obtained results for the equipartitions with non-parallel hyperplanes. (See [4] and [5].)

The paper also serves as the illustration of the applicability and the power of the methods of equivariant topology (more precisely, equivariant index theory) in the problems of geometric combinatorics.

1 Introduction

Any collection of k hyperplanes in $\mathbb{R}^d$ determine a partition of this Euclidean space (and any mass distribution in it) into 2^k hyperorthants (defined as the intersections of the appropriate half-spaces). Given a family of j mass distributions in $\mathbb{R}^d$, we say that a collection of k hyperplanes forms a equipartition of these j mass distributions if each hyperorthant contains exactly $\frac{1}{2^k}$ of each of the given mass distributions.

The question when every family of j mass distributions in $\mathbb{R}^d$ admit an equipartition by some collection of k hyperplanes, is known as the equipartition problem, and it was formulated by B. Grünbaum in 1960 (see [2]).

It attracted a lot of attention and some answers to this problem are already obtained in [3]. Very thorough treatment of this question is presented in [5], where more complete results are obtained. However, the question remains unsettled in general, and is still considered as an important and difficult question in the area of discrete and computational geometry/topology. The most recent and the most

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complete answers to this question are given in [4]. The most important open question is whether every mass distribution in $\mathbb{R}^4$ admits an equipartition by 4 hyperplanes in 16 hyperorthants.

In this paper we treat the related problem of equipartition of a mass distribution by a family of hyperplanes in a special position, namely by a collection of parallel hyperplanes and one or more additional non-parallel hyperplanes. Since such a collection of hyperplanes divides $\mathbb{R}^d$ in the box-like regions (or boxes), we will refer to this question as to the question of equipartition of a mass distribution in boxes.

We obtain the general sufficient condition on the dimension d and the number of parallel hyperplanes so that every mass distribution in $\mathbb{R}^d$ admits such equipartition. (See theorems 3.1, 4.1, 4.5, 5.1 and 5.2.) As the sample results illustrating the obtained results we mention here the following three (see corollary 3.2, corollary 4.3 and corollary 4.4).

Claim 1. Every mass distribution in the plane admits an equipartition in 6 boxes by two parallel lines and one additional line not parallel to them.

Claim 2. Any mass distribution in $\mathbb{R}^4$ could be equipartitioned in $12 = 3 \times 2 \times 2$ boxes by a collection of 4 hyperplanes two of which are parallel.

Claim 3. Any mass distribution in $\mathbb{R}^8$ could be equipartitioned in $7 \times 2 \times 2$ boxes by a collection of 6 parallel hyperplanes and two additional non-parallel hyperplanes.

Notice that the claim 2 is related to the (above mentioned) most important open case of the original question of B. Grünbaum. We don't know whether for every mass distribution in $\mathbb{R}^4$, there exists a 4-tuple of hyperplanes equipartitioning this mass distribution in 16 hyperorthants. But, if we consider 4-tuples of hyperplanes two of which are parallel, they divide the space and the mass distribution in 12 boxes, and claim 2 shows that we could always find such a 4-tuple equipartitioning the measure.

For technical reasons, we treat separately the cases of even and odd number of parallel hyperplanes. We first discuss the case of even number of parallel hyperplanes, and then explain the differences in the formulation and the proof of the odd case.

Throughout this paper, we work with the continuous mass distributions with the positive measure of any open set in $\mathbb{R}^d$. (A continuous mass distribution is a finite Borel measure μ defined by the formula $\mu(A) = \int_A f\, d\mu$ for an integrable density function $f : \mathbb{R}^d \to R$.) Because of that, the hyperplane orthogonal to some direction and partitioning the given mass distribution in the given ratio is unique. Using the limit argument, it is easy to extend the result to all mass distributions which are weak limits of the mass distributions satisfying the above properties. In particular, the results are true for measurable sets and for finitely supported measures.

We use the topological method in treating this question. More precisely, we reduce the above question to the question of the existence of an equivariant map. There is a number of ways to treat the latter question, such as the use of characteristic classes or the use of the obstruction theory. We find it most convenient to use the index theory approach as formulated by E. Fadell and S. Husseini in [1].

The application of topological methods in combinatorics dates back (at least) to 70's and the papers by L. Lovász, I. Bárány and others. The appearance of these ideas and their development served as the starting point in the creation of the new subfield, topological combinatorics.

2 A short review of index theory

For the reader's convenience, we present a very short review of the ideal-valued cohomological index theory by E. Fadell and S. Husseini. Given a finite group G, and a G-map $f : X \to Y$ between G-spaces X and Y, we could map these spaces to the one-point space $\{*\}$ and obtain a commutative diagram of G-spaces and G-maps. Multiplying by the total space EG of the universal G-bundle $EG \to BG$, we obtain new commutative diagram of G-spaces and G-maps. (We consider the diagonal G-action on the product spaces.) Passing to the spaces of orbits, we obtain the following commutative diagram of continuous maps:

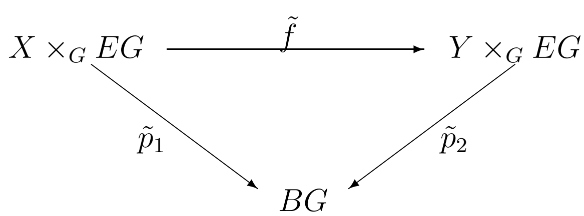

Figure 1:

which induces the following commutative diagram in cohomology:

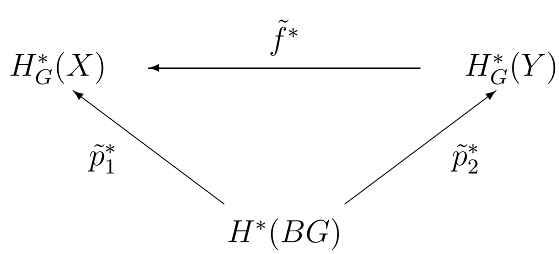

Figure 2:

The kernels of the maps $\tilde{p}^*_1$ and $\tilde{p}^*_2$ are the ideals in the cohomology ring of the classifying space of the group G, and they are called indices and denoted by $\mathrm{Ind}_G X$ and $\mathrm{Ind}_G Y$ respectively.

The commutativity of the above diagram implies the relation $\mathrm{Ind}_G Y \subseteq \mathrm{Ind}_G X$. If we could prove that this inclusion relation is not satisfied, we would obtain a contradiction proving that a G-equivariant map $f : X \to Y$ does not exist.

We refer the reader to the original paper [1] for additional properties of the index and some basic computation. Some other computations, needed in this paper could be found in [6], Corollary 2.12 and Proposition 2.7.

3 The problem and the results - 2 directions

In order to describe the method and develop the intuition in a more acceptable way, we choose to treat the simplest particular case first.

So, in this section we treat the question of equipartition of a mass distribution in $\mathbb{R}^d$ by a collection of parallel hyperplanes and by one additional hyperplane (not parallel to them). We show that the greatest number of parallel hyperplanes for which such an equipartition always exists (for every mass distribution in $\mathbb{R}^d$) is $2d-2$. The same result will be true also for $2d-3$ parallel hyperplanes in $\mathbb{R}^d$, but the equipartition with $2d-1$ parallel hyperplanes are always possible only in $\mathbb{R}^{d+1}$. The obtained result is the best possible in the sense that for greater number of hyperplanes, the corresponding equivariant mapping exists.

Theorem 3.1 *For every mass distribution in $\mathbb{R}^d$ there is a collection of $2d-2$ parallel hyperplanes and one additional hyperplane dividing $\mathbb{R}^d$ in $4d-2$ boxes containing the same amount of the mass distribution.*

Especially, when $d = 2$, we obtain the proof of the following corollary.

Corollary 3.2 *Every mass distribution in the plane admits an equipartition in 6 boxes by two parallel lines and one additional line not parallel to them.* ■

The proof of the theorem 3.1 (with the complete description of the approach, needed also in the proofs of other results from this paper) is contained in the following two subsections.

3.1 Reduction

In this subsection we reduce the statement of the above theorem to the topological statement.

For any mass distribution in $\mathbb{R}^d$ and any pair of vectors $(u, v) \in S^{d-1} \times S^{d-1}$, let $H_1^u, H_2^u, ..., H_{2d-2}^u$ be the oriented hyperplanes orthogonal to u, ordered in the direction of the vector u, and dividing $\mathbb{R}^d$ into $2d-1$ regions each containing the same amount (i.e. $\frac{1}{2d-1}$) of the considered mass distribution. Also, let H^v be the oriented hyperplane orthogonal to v dissecting a mass distribution into two halfspaces containing the same amount of the mass distribution.

These hyperplanes form $2(2d-1)$ boxes and the measure of these boxes form a $2 \times (2d-1)$ matrix of the form

$$\begin{pmatrix} \alpha_0 + \alpha_1 & \alpha_0 + \alpha_2 & \dots & \alpha_0 + \alpha_{2d-1} \\ \alpha_0 - \alpha_1 & \alpha_0 - \alpha_2 & \dots & \alpha_0 - \alpha_{2d-1} \end{pmatrix}$$

where $\alpha_0 = \frac{1}{2(2d-1)}$ and $\alpha_1 + \alpha_2 + \dots + \alpha_{2d-1} = 0$.

So, we could identify the configuration space of our problem to be the product of two spheres $S^{d-1} \times S^{d-1}$ and the test space as the space of all $2 \times (2d-1)$ matrices of the above form. The group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ acts naturally both on configuration space and the test space (by the obvious permutations). The test space could also be seen as the $(2d-2)$-dimensional linear representation of the group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, which we denote by V. The test map $f : S^{d-1} \times S^{d-1} \to V$, which maps each pair of unit vectors to the measures of the corresponding boxes, is easily seen to be $(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$-equivariant.

Now, our problem is reduced to the topological claim that the matrix

$$\begin{pmatrix} \alpha_0 & \alpha_0 & \dots & \alpha_0 \\ \alpha_0 & \alpha_0 & \dots & \alpha_0 \end{pmatrix}$$

(obtained when $\alpha_1 = \dots = \alpha_{2d-1} = 0$) belongs to the image of the test map f. Suppose, to the contrary, this not to be the case. Then we would have a $(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$-equivariant map $f : S^{d-1} \times S^{d-1} \to S(V)$, where $S(V)$ denotes the unit sphere of the representation space V. Finally, we reach a contradiction (proving in this way our claim), by showing that such equivariant map with the actions of our group described formerly could not exist. In proving this we use the ideal valued cohomological index theory of Fadell and Husseini.

3.2 Computation

We will use the approach described above to show that there is no $(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$-equivariant map $f : S^{d-1} \times S^{d-1} \to S(V)$, where $S(V)$ denotes the unit sphere of the representation space V described in the section 3.1. So, we work with the group $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2$, and it is well known that $BG = B\mathbb{Z}/2 \times B\mathbb{Z}/2 = \mathbb{R}P^\infty \times \mathbb{R}P^\infty$, and $H^*(BG; \mathbb{Z}/2) \cong \mathbb{Z}/2[x, y]$, where x and y are the free generators of this polynomial ring in dimension 1 both.

The generators of the group $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ act by the antipodal action on the corresponding spheres in $S^{d-1} \times S^{d-1}$. It is well known that in this case we have $\mathrm{Ind}_G(S^{d-1} \times S^{d-1}) = (x^d, y^d)$, i.e. the index is the ideal generated by the monomials x^d and y^d.

Now we determine the index of the unit sphere in the representation space V. We refer the reader to the survey article [6] (Corollary 2.12), for the necessary background for the following computation. As we noticed in the section 3.1, V is the $(2d-2)$-dimensional representation which could be described as the space of all $2 \times (2d-1)$

matrices of the form

$$\begin{pmatrix} \alpha_0+\alpha_1 & \alpha_0+\alpha_2 & \dots & \alpha_0+\alpha_{2d-1} \\ \alpha_0-\alpha_1 & \alpha_0-\alpha_2 & \dots & \alpha_0-\alpha_{2d-1} \end{pmatrix}$$

where $\alpha_0 = \frac{1}{2(2d-1)}$ and $\alpha_1+\alpha_2+\cdots+\alpha_{2d-1} = 0$. The generator of the first copy of $\mathbb{Z}/2$ acts on such matrices by permuting the columns in the reverse order, i.e. by sending $(\alpha_1, \alpha_2, ..., \alpha_{2d-1})$ to $(\alpha_{2d-1}, \alpha_{2d-2}, ..., \alpha_1)$. The generator of the second copy of $\mathbb{Z}/2$ acts by permuting two rows, i.e. by sending each α_i to $-\alpha_i$. By the relation $\alpha_1+\alpha_2+\cdots+\alpha_{2d-1} = 0$, the element α_d is determined by the remaining elements. To shorten the notation we will subtract α_0 from the entries of the above mentioned matrix, and present the matrix in the form

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{2d-1} \\ -\alpha_1 & -\alpha_2 & \dots & -\alpha_{2d-1} \end{pmatrix}$$

So, V is a $(2d-2)$-dimensional representation of the group $\mathbb{Z}/2\oplus\mathbb{Z}/2$. The representation space V splits in the sum of $2d-2$ invariant 1-dimensional representations. We present $d-1$ pairs of this G-invariant 1-dimensional representations. The i-th pair of this representations form the matrices of the form:

$$\begin{pmatrix} \dots & \alpha & \dots & -2\alpha & \dots & \alpha & \dots \\ \dots & -\alpha & \dots & 2\alpha & \dots & -\alpha & \dots \end{pmatrix}$$

and the matrices of the form:

$$\begin{pmatrix} \dots & \alpha & \dots & 0 & \dots & -\alpha & \dots \\ \dots & -\alpha & \dots & 0 & \dots & \alpha & \dots \end{pmatrix}$$

Here, we write only the entries in the i-th, d-th, and $(2d-i)$-th column of the matrix, while all other entries are 0.

The generator of the first copy of $\mathbb{Z}/2$ acts on such matrices by permuting the columns in the reverse order, and so it acts trivially on the first mentioned 1-dimensional subspace of matrices and antipodally on the second. The generator of the second copy of $\mathbb{Z}/2$ acts by permuting two rows, and so it acts antipodally on both 1 dimensional subspaces of matrices.

So, the index $\mathrm{Ind}_G S(V)$ is the ideal in the polynomial ring $\mathbb{Z}/2[x,y]$ generated by the polynomial $(y(x+y))^{d-1}$. (Consult [6].)

All the summands of the polynomial

$$y^{d-1}(x+y)^{d-1} = x^{d-1}y^{d-1} + \binom{d-1}{1}x^{d-2}y^d + \cdots + y^{2d-2},$$

belong to the ideal (x^d, y^d), except for the first one $x^{d-1}y^{d-1}$. So, this polynomial is not contained in the ideal (x^d, y^d). This means that $\mathrm{Ind}_G S(V) \nsubseteq \mathrm{Ind}_G(S^{d-1}\times S^{d-1})$.

The considerations from the previous subsection show that there is no $(\mathbb{Z}/2 \oplus \mathbb{Z}/2)$-equivariant mapping from $S^{d-1} \times S^{d-1}$ to $S(V)$, which implies that every equivariant map from $S^{d-1} \times S^{d-1}$ to the representation space V maps some pair of unit vectors (u, v) to the matrix

$$\begin{pmatrix} \alpha_0 & \alpha_0 & \dots & \alpha_0 \\ \alpha_0 & \alpha_0 & \dots & \alpha_0 \end{pmatrix}$$

This completes the argument and proves our theorem. ■

If we consider the case of odd number of parallel hyperplanes ($2d-1$ of them), the similar considerations would prove the following theorem.

Theorem 3.3 *For every mass distribution in $\mathbb{R}^{d+1}$ there is a collection of $2d-1$ parallel hyperplanes and one additional hyperplane dividing it in $4d$ boxes containing the same amount of the mass distribution.* ■

The major difference is that we are now faced with two central columns, since the matrix has even number of columns ($2d$ of them), and in very similar way we get that the index of the sphere in the representation space is the ideal generated by the polynomial $y^{d-1}(x+y)^d$. This polynomial belongs to the ideal generated by monomials x^d and y^d, but does not belong to the ideal generated by monomials x^{d+1} and y^{d+1}. Notice that in the case $m = 2$ the stronger result (in some sense) is obtained for even number of hyperplanes. Namely, any mass distribution in $\mathbb{R}^{d+1}$ could be equipartitioned also in $4d + 2$ boxes by some $2d$ parallel hyperplanes and one additional non-parallel to them.

4 The case of 3 directions

In this section we generalize the result from the previous section to the case of equipartition of a mass distribution in some Euclidean space $\mathbb{R}^d$ by a collection of parallel hyperplanes orthogonal to the direction u, and by two additional hyperplanes not parallel neither to the first mentioned collection nor one to each other. First we consider even number $2k$ of parallel hyperplanes. In this case we treat the equipartition of a mass distribution in $(2k+1) \times 2 \times 2$ boxes. Our aim is to determine the sufficient condition on the dimension d and the number k so that every mass distribution in $\mathbb{R}^d$ admits such an equipartition.

Since we use the index theory again, we provide an algorithm to decide the above question for a pair of numbers d and k, which reduces the question to the question whether some polynomial belongs to some ideal in the polynomial algebra $\mathbb{Z}/2[x_1, x_2, x_3]$ over 3 variables.

Let us denote with $\mathbb{P}_3(x_1, x_2, x_3) = x_1 x_2 x_3 (x_1 + x_2)(x_1 + x_3)(x_2 + x_3)(x_1 + x_2 + x_3)$ the Dickson polynomial in 3 variables. Over $\mathbb{Z}/2$, this Dickson polynomial has also another description $\mathbb{P}_3(x_1, x_2, x_3) = \sum_{\sigma \in S_3} x^4_{\sigma(1)} x^2_{\sigma(2)} x_{\sigma(3)}$, and as a consequence we get:

Theorem 4.1 *Let*

$$\mathbb{P}_3 = \mathrm{Det}\begin{bmatrix} x_1 & x_1^2 & x_1^4 \\ x_2 & x_2^2 & x_2^4 \\ x_3 & x_3^2 & x_3^4 \end{bmatrix} \in \mathbb{Z}/2[x_1, x_2, x_3]$$

be a Dickson polynomial. Then every measure in $\mathbb{R}^d$ admits an equipartition by a collection of $2k$ parallel hyperplanes and two additional non-parallel hyperplanes in $(2k+1)\times 2\times 2$ boxes if

$$(x_2 + x_3)\left(\frac{1}{x_1}\mathbb{P}_3\right)^k \notin (x_1^d, x_2^d, x_3^d).$$

Proof: Again, for any mass distribution in $\mathbb{R}^d$ and any triple of vectors $(u, v, w) \in S^{d-1}\times S^{d-1}\times S^{d-1}$, let $H_1^u, H_2^u, ..., H_{2k}^u$ be the oriented hyperplanes orthogonal to u, ordered in the direction of the vector u, and dividing $\mathbb{R}^d$ into $2k+1$ regions each containing the same amount (i.e. $\frac{1}{2k+1}$) of the considered mass distribution. Also, let H^v and H^w be the oriented hyperplanes orthogonal to v and w respectively, each dissecting a mass distribution into two halfspaces containing the same amount of the mass distribution.

These hyperplanes form $(2k+1)\times 2\times 2$ boxes and the measure of these boxes form a 3-dimensional $(2k+1)\times 2\times 2$ matrix. We describe this matrix by its $2k+1$ two-dimensional "slices" which are 2×2 matrices, and are of the form

$$\begin{pmatrix} \varrho + \alpha_i & \varrho + \beta_i \\ \varrho + \gamma_i & \varrho + \delta_i \end{pmatrix}$$

$(i = 1, 2, ..., 2k+1)$, where $\varrho = \frac{1}{4(2k+1)}$, and $\alpha_i + \beta_i + \gamma_i + \delta_i = 0$ for every $i = 1, 2, ..., 2k+1$. Also, the entries of this 3-dimensional matrix satisfy two additional relations (coming from the properties of the hyperplanes H^v and H^w), and those are $\sum_i(\alpha_i + \beta_i) = 0$ and $\sum_i(\alpha_i + \gamma_i) = 0$.

In this case the configuration space of our problem is the product of three spheres $S^{d-1}\times S^{d-1}\times S^{d-1}$ and the test space is the space of all $(2k+1)\times 2\times 2$ matrices of the above form. The group $\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$ acts naturally both on configuration space and the test space. So, equivalently the test space could be represented as a $(6k+1)$-dimensional linear representation V of the group $\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$.

The test map $f : S^{d-1}\times S^{d-1}\times S^{d-1} \to V$, which maps each triple of unit vectors to the measures of the corresponding boxes, is easily seen to be $(\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2)$-equivariant.

Now, our problem is reduced to the topological claim that the matrix with all entries equal to $\varrho = \frac{1}{4(2k+1)}$ (obtained when $\alpha_i = \beta_i = \gamma_i = \delta_i = 0$ for every $i = 1, 2, ..., 2k+1$) belongs to the image of the test map f. Suppose, to the contrary, this not to be the case. Then we would have a $(\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2)$-equivariant map $f : S^{d-1} \times S^{d-1} \times S^{d-1} \to S(V)$, where $S(V)$ denotes the unit sphere of the representation space V.

As in the previous case, we reach a contradiction (proving in this way our claim), by showing that such equivariant map with the described actions of our group could not exist.

In this case we have the group $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$, and it is well known that $H^*(BG; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_1, x_2, x_3]$, where x_1, x_2, and x_3 are the free generators of this polynomial ring in dimension 1 all.

The generators of the group G act by the antipodal action on the corresponding spheres in $S^{d-1} \times S^{d-1} \times S^{d-1}$. It is well known that in this case we have $\mathrm{Ind}_G(S^{d-1} \times S^{d-1} \times S^{d-1}) = (x_1^d, x_2^d, x_3^d)$, i.e. the index is the ideal generated by the monomials x_1^d, x_2^d, and x_3^d.

Now we determine the index of the unit sphere in the representation space V. The generator of the first copy of $\mathbb{Z}/2$ permutes the "slices" of the matrix in the reverse order, i.e. by sending $\alpha_i, \beta_i, \gamma_i, \delta_i$ to $\alpha_{2k+2-i}, \beta_{2k+2-i}, \gamma_{2k+2-i}, \delta_{2k+2-i}$. The generator of the second copy of $\mathbb{Z}/2$ acts by permuting two 2-dimensional "rows" of the matrix, i.e. by sending each α_i and β_i to γ_i and δ_i respectively. The generator of the third copy of $\mathbb{Z}/2$ also acts by permuting two 2-dimensional "rows" of the matrix, i.e. by sending each α_i and γ_i to β_i and δ_i respectively.

To shorten the notation we will subtract ϱ from the entries of the above mentioned matrix, and present the "slices" of the matrix in the form

$$\begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$$

The representation space V splits in the sum of $6k+1$ G-invariant 1-dimensional representations. We present here k 6-tuples of this G-invariant 1-dimensional representations and additionally the last one. The i-th 6-tuple of this representations form the matrices whose i-th, $(k+1)$-th, and $(2k+2-i)$-th "slices" are of the following forms (the "slices" will be separated by the vertical lines, remember that the entries in the remaining "slices" are all 0):

$$\left(\cdots \left| \begin{matrix} \lambda & \lambda \\ -\lambda & -\lambda \end{matrix} \right| \cdots \left| \begin{matrix} -2\lambda & -2\lambda \\ 2\lambda & 2\lambda \end{matrix} \right| \cdots \left| \begin{matrix} \lambda & \lambda \\ -\lambda & -\lambda \end{matrix} \right| \cdots \right)$$

$$\left(\cdots \left| \begin{matrix} \lambda & \lambda \\ -\lambda & -\lambda \end{matrix} \right| \cdots \left| \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \right| \cdots \left| \begin{matrix} -\lambda & -\lambda \\ \lambda & \lambda \end{matrix} \right| \cdots \right)$$

$$\left(\cdots \left| \begin{matrix} \lambda & -\lambda \\ \lambda & -\lambda \end{matrix} \right| \cdots \left| \begin{matrix} -2\lambda & 2\lambda \\ -2\lambda & 2\lambda \end{matrix} \right| \cdots \left| \begin{matrix} \lambda & -\lambda \\ \lambda & -\lambda \end{matrix} \right| \cdots \right)$$

$$\left(\cdots \left| \begin{matrix} \lambda & -\lambda \\ \lambda & -\lambda \end{matrix} \right| \cdots \left| \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \right| \cdots \left| \begin{matrix} -\lambda & \lambda \\ -\lambda & \lambda \end{matrix} \right| \cdots \right)$$

$$\left(\cdots \left| \begin{matrix} \lambda & -\lambda \\ -\lambda & \lambda \end{matrix} \right| \cdots \left| \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \right| \cdots \left| \begin{matrix} \lambda & -\lambda \\ -\lambda & \lambda \end{matrix} \right| \cdots \right)$$

$$\left(\cdots \left| \begin{array}{cc} \lambda & -\lambda \\ -\lambda & \lambda \end{array} \right| \cdots \left| \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right| \cdots \left| \begin{array}{cc} -\lambda & \lambda \\ \lambda & -\lambda \end{array} \right| \cdots \right)$$

The additional invariant 1-dimensional representation have non-zero entries only in the $(k+1)$-th "slice" and is of the form

$$\left(\cdots \left| \begin{array}{cc} \lambda & -\lambda \\ -\lambda & \lambda \end{array} \right| \cdots \right)$$

According to the described action of the generators of the group G on these 3-dimensional matrices, we see that the indices of the 1-dimensional representations from the 6-tuples are generated by the polynomials x_2, x_1+x_2, x_3, x_1+x_3, x_2+x_3, $x_1+x_2+x_3$ (in this order), and the index of the additional representation is x_2+x_3. So, by [6] (Corollary 2.12), the index of the test space $\mathrm{Ind}_G S(V)$ is the ideal in the polynomial ring $\mathbb{Z}/2[x_1,x_2,x_3]$ generated by the polynomial

$$(x_2(x_1+x_2)x_3(x_1+x_3)(x_2+x_3)(x_1+x_2+x_3))^k\,(x_2+x_3) = (x_2+x_3)\left(\frac{1}{x_1}\mathbb{P}_3\right)^k$$

and the result follows. ■

Since the polynomial $(x_2+x_3)\left(\frac{1}{x_1}\mathbb{P}_3\right)^k$ is of degree $6k+1$, it is obvious that the necessary condition for the algebraic condition in the statement of Theorem 4.1 to be fulfilled is $6k+1 \leq 3(d-1)$, which gives $d \geq 2k+2$. The sufficient condition for the same requirement is more interesting, because it determines the dimension (maybe not the smallest possible) of the Euclidean space for which the equipartition by the given number of hyperplanes always exists (for every measurable set).

In the general case (for every k), we show that the condition $d \geq 3k+1$ suffices to guarantee the relation from Theorem 4.1, implying in this way the following result.

Corollary 4.2 *Any mass distribution in* $\mathbb{R}^{3k+1}$ *could be equipartitioned in* $(2k+1)\times 2\times 2$ *boxes by a collection of* $2k$ *parallel hyperplanes and two additional non-parallel hyperplanes.*

Proof: It is easy to see that the coefficient in the polynomial $(x_2+x_3)\left(\frac{1}{x_1}\mathbb{P}_3\right)^3$ multiplying the monomial $x_2(x_1^3x_2^2x_3)^k = x_1^{3k}x_2^{2k+1}x_3^k$ equals 1. ■

It should be noticed that this estimate could not be improved (using our methods) when k is a power of 2.

However, for some other values of k the better (smaller) estimate is obtained, due to the properties of the binomial coefficients over $\mathbb{Z}/2$. Especially, when k is a little bit smaller than some power of 2, our method provides the estimate which is quite close (or even equal) to the estimate from the necessary condition $d \geq 2k+2$. We believe that this estimate could not be improved, at least in the sense that for the smaller values of dimension d, the considered equivariant mapping exists.

We illustrate the above remarks by showing that for $k = 1$ we get $d = 4$, and for $k = 3$ we get $d = 8$.

Corollary 4.3 *Any mass distribution in* $\mathbb{R}^4$ *could be equipartitioned in* $12 = 3\times 2\times 2$ *boxes by a collection of* 4 *hyperplanes two of which are parallel.* ■

Corollary 4.4 *Any mass distribution in* $\mathbb{R}^8$ *could be equipartitioned in* $7\times 2\times 2$ *boxes by a collection of* 6 *parallel hyperplanes and two additional non-parallel hyperplanes.*

Proof: We will show that the coefficients in the polynomial $(x_2 + x_3)\left(\frac{1}{x_1}\mathbb{P}_3\right)^3$ multiplying the monomials $x_1^7x_2^7x_3^5$ and $x_1^7x_2^5x_3^7$ are non-trivial. Since these monomials do not belong to the ideal generated by the monomials x_1^8, x_2^8 and x_3^8, the corollary follows.

The third power of the sum of some monomials (over $\mathbb{Z}/2$) has non-zero coefficient multiplying the third power of the monomials and the product of the square of some monomial with some other monomial. It is easy to verify that there is only one way to get $x_1^7x_2^7x_3^5$ (e.g.) in the expression $(x_2+x_3)\left(\frac{1}{x_1}\mathbb{P}_3\right)^3$ and that is $x_2\cdot\left(\frac{1}{x_1}\right)^3\cdot(x_1^4x_3^2x_2)^2\cdot(x_2^4x_1^2x_3)$. So the coefficient multiplying $x_1^7x_2^7x_3^5$ is non-zero, and we are done. ■

Let us now turn to the case od odd number od parallel hyperplanes and two additional hyperplanes. In almost the same way we obtain:

Theorem 4.5 *Let*

$$\mathbb{P}_3 = \text{Det}\begin{bmatrix} x_1 & x_1^2 & x_1^4 \\ x_2 & x_2^2 & x_2^4 \\ x_3 & x_3^2 & x_3^4 \end{bmatrix} \in \mathbb{Z}/2[x_1, x_2, x_3]$$

be a Dickson polynomial. Then every measure in $\mathbb{R}^d$ *admits an equipartition by a collection of* $2k+1$ *parallel hyperplanes and two additional non-parallel hyperplanes in* $(2k+2)\times 2\times 2$ *boxes if*

$$\frac{1}{x_2x_3}\left(\frac{1}{x_1}\mathbb{P}_3\right)^{k+1} \notin (x_1^d, x_2^d, x_3^d).$$

Proof: Let us only explain the differences. We have the 3-dimensional matrix with even number of slices and we have the central **pair** of slices. This central pair forms a $2\times 2\times 2$ matrix subject to 4 relations. The corresponding representation space splits in the sum of 4 one-dimensional representations, those from the 6-tuple in the proof of theorem 4.1 having only zeros in the central slice. Therefore, the corresponding index of this representation space is:

$$\left(\frac{1}{x_1}\mathbb{P}_3\right)^k \cdot (x_1+x_2)(x_1+x_3)(x_2+x_3)(x_1+x_2+x_3) = \frac{1}{x_2x_3}\left(\frac{1}{x_1}\mathbb{P}_3\right)^{k+1}$$

The result follows. ■

Similarly as for even number of parallel hyperplanes, we have a necessary condition and a sufficient condition on numbers d and k, to satisfy the relation from the above theorem.

Since the polynomial $\frac{1}{x_2x_3}\left(\frac{1}{x_1}\mathbb{P}_3\right)^{k+1}$ is of degree $6k+4$, it is obvious that the necessary condition is $6k+4 \leq 3(d-1)$, which gives $d \geq 2k+3$.

In similar way as above we could show that the condition $d \geq 3k+4$ is sufficient to guarantee the relation from Theorem 4.5, and to imply the following.

Corollary 4.6 *Any mass distribution in* $\mathbb{R}^{3k+4}$ *could be equipartitioned in* $(2k+2)\times 2\times 2$ *boxes by a collection of* $2k+1$ *parallel hyperplanes and two additional non-parallel hyperplanes.*

Proof: Now the coefficient in the polynomial $\frac{1}{x_2x_3}\left(\frac{1}{x_1}\mathbb{P}_3\right)^{k+1}$ multiplying the monomial $\frac{1}{x_2x_3}(x_1^3x_2^2x_3)^{k+1} = x_1^{3k+3}x_2^{2k+1}x_3^k$ equals 1. ■

Again, it should be noticed that this estimate could not be improved by our methods when $k+1$ is a power of 2, but for some other values of k a better estimate could be obtained. When $k+1$ is a little bit smaller than some power of 2, we believe that the obvious necessary condition $d \geq 2k+3$ is much closer to the best possible estimate.

These estimates could be presented in a unified way, by saying that for l parallel hyperplanes (l being even or odd), a necessary condition is $d \geq l+2$ and a sufficient condition is $d \geq 3[\frac{l+1}{2}]+1$.

An easy algebraic calculation provides us with the following table in which we describe, for small numbers l (being even or odd) of parallel hyperplanes, the smallest dimension d (provided by our methods) of the Euclidean space in which the equipartition with that many parallel hyperplanes and two additional non-parallel to them, is always possible. Notice that for $l = 2^j$ and for $l = 2^j - 1$ we have $d = 3[\frac{l+1}{2}]+1$, and for $l = 2^j - 2$ we have $d = l+2$.

l	2	3 - 4	5 - 6	7 - 10	11 - 12	13 - 14	15 - 22
d	4	7	8	13	15	16	25

Reading this table in the other direction, we see that in $\mathbb{R}^4$ (and in $\mathbb{R}^5$ and $\mathbb{R}^6$) the equipartition is always possible with 2 parallel hyperplanes (and two non-parallel, which we do not mention any further), in $\mathbb{R}^7$ with 4, in $\mathbb{R}^8$ (and up to $\mathbb{R}^{12}$) with 6, in $\mathbb{R}^{13}$ (and in $\mathbb{R}^{14}$) with 10, in $\mathbb{R}^{15}$ with 12, in $\mathbb{R}^{16}$ (and up to $\mathbb{R}^{24}$) with 14, in $\mathbb{R}^{25}$ with 22, and so on.

Notice that again for $l = 14$ we get $d = 16$ which is the best possible in the same sense as above. Notice also that in all these examples, due to the arithmetic reasons, the resulting Euclidean space has the same dimension for the odd number $2k-1$ of parallel hyperplanes and for the next even number $2k$ of parallel hyperplanes.

5 The general case

It is obvious how to generalize these statements to the case of more than 3 directions, to obtain the complete algorithm for the determination of the dimension d so that any mass distribution in $\mathbb{R}^d$ admits an equipartition in boxes. Without going much into details, we formulate the result obtained for the case of m directions and sketch the proof. The reader could easily modify the above argument to fill in the details.

Again, we formulate two separate statements, one for the case of even and the other for the case of odd number of parallel hyperplanes. Similarly to the previous case, with $\mathbb{P}_m(x_1, ..., x_m)$ we denote the Dickson polynomial in m variables. Again, it is the product of all linear combinations of these variables. Over $\mathbb{Z}/2$ it could also be described by $\mathbb{P}_m(x_1, ..., x_m) = \sum_{\sigma \in S_m} x_{\sigma(1)}^{2^{m-1}} \cdots x_{\sigma(m)}$. The Dickson polynomial $\mathbb{P}_{m-1}$ mentioned below will be in $m-1$ variables $x_2, ..., x_m$.

Theorem 5.1 *Let*

$$\mathbb{P}_m = \mathrm{Det}\begin{bmatrix} x_1 & x_1^2 & \dots & x_1^{2^{m-1}} \\ x_2 & x_2^2 & \dots & x_2^{2^{m-1}} \\ \vdots & \vdots & \ddots & \vdots \\ x_m & x_m^2 & \dots & x_m^{2^{m-1}} \end{bmatrix} \in \mathbb{Z}/2[x_1, x_2, ..., x_m]$$

be a Dickson polynomial. Then every measure in $\mathbb{R}^d$ admits an equipartition by a collection of $2k$ parallel hyperplanes and $m-1$ additional non-parallel hyperplanes in $(2k+1) \times 2 \times \cdots \times 2$ boxes if

$$\frac{1}{x_2 x_3 \cdots x_m}\mathbb{P}_{m-1}(x_2, ..., x_m)\left(\frac{1}{x_1}\mathbb{P}_m\right)^k \notin (x_1^d, x_2^d, ..., x_m^d).$$

Proof: The proof goes along exactly the same lines as the proof of Theorem 4.1. Here the statement is reduced to the non-existence of $(\mathbb{Z}/2)^{\oplus m}$-equivariant map from the configuration space $(S^{d-1})^m$ to the unit sphere in the representation space V, which is $(k(2^m-2) + 2^{m-1} - m)$-dimensional.

Here, the cohomology of the classifying space of the group $G = (\mathbb{Z}/2)^{\oplus m}$ is the polynomial algebra over m generators and the index of the configuration space is the ideal generated by monomials $x_1^d, x_2^d, ..., x_m^d$.

In order to split the representation space in G-invariant 1-dimensional representations, we consider again the pairs of strips determined by parallel hyperplanes and the central strip. We see similarly that each pair of strips is split into $2(2^{m-1}-1) = 2^m - 2$ invariant 1-dimensional representations. Determining the generators of G which act non-trivially on each of these 1-dimensional representations, we see that their indices are all homogeneous degree 1 polynomials (over $\mathbb{Z}/2$) in m variables $x_1, x_2, ..., x_m$ except for the monomial x_1. The central strip is split into $2^{m-1} - m$ invariant 1-dimensional representations whose indices are all homogeneous degree 1 polynomials in $m-1$ variables $x_2, ..., x_m$ except for the monomials $x_2, x_3, ..., x_m$. The result follows. ■

Since the above polynomial is of degree $k(2^m-2)+2^{m-1}-m$, we obtain similarly as in the case $m = 3$ the obvious necessary condition $k(2^m-2)+2^{m-1}-m \leq m(d-1)$. A sufficient condition is $(2^{m-1}-1)k \leq d-1$ and it could not be improved when k is a power of 2. It says that in the Euclidean space of dimension $d = (2^{m-1}-1)k+1$ the equipartition with $2k$ parallel hyperplanes and $m-1$ additional hyperplanes is always possible.

In the case of odd number of parallel hyperplanes, we have the following statement.

Theorem 5.2 *Let*

$$\mathbb{P}_m = \mathrm{Det} \begin{bmatrix} x_1 & x_1^2 & \dots & x_1^{2^{m-1}} \\ x_2 & x_2^2 & \dots & x_2^{2^{m-1}} \\ \vdots & \vdots & \ddots & \vdots \\ x_m & x_m^2 & \dots & x_m^{2^{m-1}} \end{bmatrix} \in \mathbb{Z}/2[x_1, x_2, ..., x_m]$$

be a Dickson polynomial. Then every measure in $\mathbb{R}^d$ *admits an equipartition by a collection of* $2k+1$ *parallel hyperplanes and* $m-1$ *additional non-parallel hyperplanes in* $(2k+2) \times 2 \times \cdots \times 2$ *boxes if*

$$\frac{1}{x_2 x_3 \cdots x_m} \left(\frac{1}{x_1} \mathbb{P}_m \right)^{k+1} \notin (x_1^d, x_2^d, ..., x_m^d).$$

■

The proof is obtained by the obvious modification of the proof of the previous theorem.

In the same way as before we see that a necessary condition is $(k+1)(2^m-2) - m + 1 \leq m(d-1)$ and a sufficient condition is $(k+1)(2^{m-1}-1) \leq d-1$.

6 Concluding remarks

6.1 Limitations of the method

Our method does not provide the answer to the case when we consider the collections of parallel hyperplanes in 2 or more directions. Namely, there are infinitely many fixed points of the action of the group G on the test space in these cases, and so the equivariant map exists. The same is true if we consider the case of more than one mass distribution.

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